Logical relations of agreement, separability, solidarity, and consistency in the Theory of Justice

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Abstract

We suggest a resource-allocation rule without any injustice that compromises between equal distribution and distributing corresponding exactly to the productivity of agents. We use priority to characterize this rule. Priority is an axiom that high-productive agent should receive less resource than the low-productive one, and should after all get more production than the low-productive. The ethic of impartiality is a fair concept that all the agents in a society are treated differently in the sense of resource-allocation, only because of their productivity. Agreement means that when productivity of some agents are changed in a society, there are no discriminating treatment for the change of the society between members of a group in which all the productivity of agents are unchanged. Consistency says that although the economy changes, there are no change for personal wealth of a group in which all the productivity of agents are unchanged if the partial given wealth of the group is not changed. We characterize resource-allocation rules that satisfy priority, and agreement, and satisfy priority, consistency, and resource continuity under the ethic of impartiality. They comprise a class of rules that are equal for some index of outcome and resource.

Keywords: Agreement, consistency, separability, priority, solidarity, allocation rules, characterization result.

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1 Introduction

In this paper, we deal with the problem of fair distribution. In any society, every agent has different ability to produce outcome from resource, so called productivity. We assume that every society has a total social wealth and a social planner so that the social planner allocates all the social wealth to the agents. Between agents, resources are never exchanged, therefore a fair distribution can only be achieved by a reasonable allocation of resources by the social planner. When we determine how to allocate the resource in the economy, we only consider the personal capability to reproduce outcome from resource, instead of all other personal attributes, which is mentioned by Moreno-Ternero and Roemer (2006) as impartiality.

Moreno-Ternero and Roemer (2006) apply two axioms named ‘priority’ and ‘solidarity’ to characterize fairness. Priority says that no individual can dominate another in both wealth and outcome. Solidarity is the principle that when new agents join an economy under a fair allocation rule, the wealth allocated to the original agents should change in the same direction. They define a function that determines the value of the pair of individual wealth and outcome. Then, they introduce index-egalitarian rule that equalizes the value of the function. Their characterization result shows that a rule combining the two principles mentioned above is equivalent to the Index-Egalitarian rule under impartiality.

Additionally, they bring out consistency and resource monotonicity. Consistency is a similar, but weaker principle than solidarity that when new agents join an economy under a fair allocation rule, and the sum of the wealth allocated to the original agents are unchanged, then the personal wealth of original agents should not be changed. Resource monotonicity is the principle that when the economy changes with same group of agents, all the agents should all gain or all lose their personal wealth by the change of economy. They also characterize the set of allocation rules satisfy priority, and consistency and resource monotonicity under impartiality.

We bring out the principle of agreement and separability. We also bring
out \textit{n-index-egalitarian rule} which is similar to \textit{index-egalitarian rule} but has a larger domain than \textit{index-egalitarian rule}. Agreement is the similar, but weaker principle than solidarity that when productivity of some agents are changed with the number of agents unchanged in an economy under a fair allocation rule, the wealth allocated to the ”unchanged” agents should change in the same direction. Separability is a weaker condition even than agreement and consistency that when productivity of some agents are changed with the number of agents unchanged in an economy under a fair allocation rule, and the sum of the wealth of ”unchanged” agents are not changed, then their personal wealth should not be changed. Additionally, we bring out Resource continuity (Moreno-Ternero and Roemer, 2006) for our characterizations. In this paper, consistency and resource continuity, and agreement are applied respectively, instead of solidarity, and it is shown that under the ethic of impartiality, a rule combining priority, consistency, and resource continuity is equivalent to the \textit{index-egalitarian rule}, and combining priority and agreement, or combining priority, separability, and resource monotonicity is equivalent to the \textit{n-index-egalitarian rule}

\section{Preliminaries}

Let \( I \) represent a population of individuals who produce an objectively measurable output from a resource called wealth. For each \( i \in I \), let \( u_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be the individual outcome function that transforms wealth into the outcome. It is assumed that, for each \( i \), \( u_i \) is continuous, strictly increasing, unbounded, and satisfies \( u_i(0) = 0 \). It is also assumed that \( \{u_i : i \in I\} \) constitutes a covering domain, i.e., the graphs of these functions cover the positive quadrant.

Let \( \mathcal{I} \) be the family of all finite subsets of \( I \). An economy \( e \) is defined as a triple \((N,u,W)\), where \( N = \{1,2,...,n\} \in \mathcal{I} \) is the set of individuals, \( u = (u_i)_{i \in N} \) is the profile of their outcome functions, and \( W \in \mathbb{R}_+ \) represents the available wealth. Let \( \xi \) be the family of all economies. An allocation rule is a function \( F \) that associates to each economy \( e = (N,u,W) \in \xi \) a
unique point $F(e) = (F_i(e))_{i \in N} \in \mathbb{R}^n_+$ such that $\sum_{i \in N} F_i(e) = W$. That is, an allocation rule indicates how to distribute the wealth available in an economy among its members.

We will use the following notations.

Let $e = (N, u, W) \in \xi$ and $F$ be an allocation rule and $N' \subseteq N$. $F_{N'}$ and $u_{N'}$ denote the projection of $F$ and $u$ respectively onto the set of coordinates that correspond to $N'$.

For $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $y = (y_1, ..., y_n) \in \mathbb{R}^n$, we write $x > y$ if $x_i > y_i$ for all $i = 1, ..., n$.

The two followings are definitions introduced by Moreno-Ternero and Roemer (2006).

Let $E(F, i, \alpha)$ be the set of economies including a fixed agent $i$ such that $F_i(e) = \alpha$.

**Definition:** $E(F, i, \alpha) = \{e \in \xi : F_i(e) = \alpha\}$.

Let $C(F, i, \alpha)$ be the set of all the pairs of wealth and its outcome included in any economies in $E(F, i, \alpha)$.

**Definition:** $C(F, i, \alpha) = \{(a, b) \in \mathbb{R}_+^2 : a = F_i(e); b = u_j(a) \text{ for some } e \in E(F, i, \alpha)\}$. $C(F, i, \alpha)$ is downward sloping if $(a, b), (a', b') \in C(F, i, \alpha)$ and $a' > a$, then $b' < b$. $C(F, i, \alpha)$ has a covering domain property if $\alpha_1 > \alpha_2$ then $C(F, i, \alpha_1)$ lies above $C(F, i, \alpha_2)$.

We introduce three more definitions for our proof.
$|e|$ denotes the cardinality of agents for $e \in \xi$.

**Definition:** $|e| = |N|$ for $e = (N, u, W)$.

$E_N(F, i, \alpha)$ is a set of economies that has the set of agents as $N$ and include a fixed agent $i$ such that $F_i(e) = \alpha$.

**Definition:** $E_N(F, i, \alpha) = \{ e \in \xi : F_i(e) = \alpha \text{ and } |e| = |N| \}$.

$C_N(F, i, \alpha)$ is a set of all the pairs of wealth and its outcome of any agent included in any economies in $E_N(F, i, \alpha)$.

**Definition:** $C_N(F, i, \alpha) = \{(a, b) \in \mathbb{R}_+^2 : a = F_i(e); b = u_j(a) \text{ for some } e \in E_N(F, i, \alpha) \}$.

3 Agreement, separability, solidarity, consistency and their logical relations

3.1 Basic Axioms

Impartiality is introduced by Moreno-Ternero and Roemer(2006), which means that we are excluding much information about individuals that we consider ethically irrelevant when we define rules on the class of economics $\xi$. Impartiality says that in deciding how to allocate the resource, we use our moral standard to ignore all personal attributes that are irrelevant to the problem at hand.

Priority(Moreno-Ternero and Roemer, 2006) requires that no agent can dominate another agent in both resources and outcome.
**Priority (PR):** For all $e = (N, u, W) \in \xi$ and all $i, j \in N$, if $F_i(e) < F_j(e)$, then $u_i(F_i(e)) > u_j(F_j(e))$.

Resource monotonicity (Roemer (1986)) says that when a bad or a good shock comes to an economy, all its members should share in the calamity or windfall.

**Resource Monotonicity (RM):** For all $e = (N, u, W) \in \xi$ and $e' = (N, u, W') \in \xi$, if $W' > W$, then $F(e') > F(e)$.

Resource Continuity (Moreno-Ternero and Roemer (2006)) says that allocation rule is continuous on the total wealth of an economy.

**Resource Continuity (RC):** For all $W$ and $W'$, if $W'$ accesses continuously to $W$, then $F(N, u, W')$ accesses continuously to $F(N, u, W)$.

### 3.2 Main axioms

Agreement was introduced by Moulin (1987) for surplus sharing problems. It requires that changes in the utility function $u$ of some members of the society should affect the other agents whose utility function has not been changed in the same direction: all strictly gain or all strictly lose or all remain the same.

**Agreement (AG):** For all $e = (N, u, W) \in \xi$ and $e' = (N, u', W') \in \xi$, and all $M \subseteq N$ such that $u_M = u'_M$, either $F_M(e) = F_M(e')$, $F_M(e) > F_M(e')$, or $F_M(e) < F_M(e')$.

Separability (Moulin (1987)) says that if for two problems, a subgroup of agents have the same utility function and the total amounts awarded to them
are the same, then the amount awarded to each agent in the subgroup should be the same.

**Separability (SE):** For all $e = (N, u, W) \in \xi$ and $e' = (N, u', W') \in \xi$, if there exists $M \subseteq N$ such that $u_M = u'_M$ and $\sum_{i \in M} F_i(e) = \sum_{i \in M} F_i(e')$, then $F_M(e) = F_M(e')$.

Solidarity (Moreno-Ternero and Roemer (2006)) says that the arrival of immigrants, whether or not accompanied by changes in the available wealth, should affect all original agents in the same direction.

**Solidarity (SL):** For all $e = (N, u, W) \in \xi$ and $e' = (N', u', W') \in \xi$, and all $M \subseteq N$ and $M \subseteq N'$ such that $u_M = u'_M$, either $F_M(e) = F_M(e')$, $F_M(e) > F_M(e')$, or $F_M(e) < F_M(e')$.

Consistency says that if a sub-group of individuals secedes with the resource allocated to it under a rule, then in the smaller economy the rule allocates the resource in the same way.

**Consistency (CY):** For all $e = (N, u, W) \in \xi$ and $e' = (N', u', W') \in \xi$, if there exists $M \subseteq N$ and $M \subseteq N'$ such that $u_M = u'_M$ and $\sum_{i \in M} F_i(e) = \sum_{i \in M} F_i(e')$, then $F_M(e) = F_M(e')$.

### 3.3 Logical relations between the axioms

Moreno-Ternero and Roemer(2006) shows that SL is equivalent to the combination of CY and RM.

We show that AG is equivalent to the combination of SE and RM.

**Proposition 1.** A rule satisfies AG if and only if it satisfies SE and RM.
Proof. First, we show that AG implies RM. Let F be a rule satisfying AG. Let \( e = (N, u, W) \) and \( e' = (N, u', W') \in \xi \) be such that \( u = u' \) and \( W < W' \). Since \( u = u' \), by AG, \( F(e) > F(e') \) or \( F(e) = F(e') \) or \( F(e) < F(e') \). Since \( W = \sum_{i \in N} F_i(e) < W' = \sum_{i \in N} F_i(e') \), \( F(e) < F(e') \). Therefore, AG implies RM.

Second, we show that AG implies SE. Let \( M \subseteq N \) such that \( u_M = u'_M \) and \( \sum_{i \in M} F_i(e) = \sum_{i \in M} F_i(e') \). By AG, \( F_M(e) > F_M(e') \) or \( F_M(e) = F_M(e') \) or \( F_M(e) < F_M(e') \). If \( F_M(e) > F_M(e') \), \( \sum_{i \in M} F_i(e) > \sum_{i \in M} F_i(e') \), and if \( F_M(e) < F_M(e') \), \( \sum_{i \in M} F_i(e) < \sum_{i \in M} F_i(e') \) which are impossible. Therefore, \( F_M(e) = F_M(e') \), the desired conclusion.

Conversely, we show that SE and RM together imply AG. Let F be a rule satisfying SE and RM. Let \( e = (N, u, W) \) and \( e' = (N, u', W') \in \xi \), and let \( M \) be such that \( M \subseteq N \) and \( u_M = u'_M \). Without loss of generality, assume that \( \sum_{i \in M} F_i(e) \geq \sum_{i \in M} F_i(e') \). If \( \sum_{i \in M} F_i(e) = \sum_{i \in M} F_i(e') \), then by SE, \( F_M(e) = F_M(e') \), so that AG holds. Now consider the case when \( \sum_{i \in M} F_i(e) > \sum_{i \in M} F_i(e') \). Let \( e^* = (N, u, W^*) \) be such that \( \sum_{i \in M} F(e^*) = \sum_{i \in M} F(e') \). Then by SE, \( F_M(e^*) = F_M(e') \). Meanwhile, since \( \sum_{i \in M} F_i(e) > \sum_{i \in M} F_i(e^*) \) and by RM, \( F_M(e) > F_M(e^*) = F_M(e') \), so that AG holds. \( \square \)

We will show that agreement is a weaker condition than solidarity, i.e., solidarity implies agreement and agreement doesn’t imply solidarity. We introduce consistency (Young(1987)) for the following discussion. Consistency says that if a subgroup of individuals secedes with the resource allocated to it under a rule, then in the smaller economy, the rule allocates the resource in the same way.

**Proposition 2.** SL implies AG.

Proof. Let \( e = (N, u, W) \in \xi \) and \( e' = (N, u', W') \in \xi \) and \( M \subseteq N \) such that \( u_M = u'_M \). Let F a rule satisfying SL. First, consider \( e^* = (M, u_M, \sum_{i \in M} F_i(e)) \in \xi \). By CY, \( F(e^*) = F_M(e) \). Next, add \(|N\setminus M| \) agents
to obtain \( e' = (N, u', W') \). By SL, \( F_M(e') > F_M(e) \) or \( F_M(e') = F_M(e) \) or \( F_M(e') < F_M(e) \). Therefore, agreement holds. \( \square \)

Now we show that AG is a strictly weaker condition by showing a counterexample of F that satisfies AG but not SL.

**Example 1.** Consider a rule \( F \) such that \( F(e) = u^{-1}(\lambda) \), where \( \lambda > 0 \) is chosen so that \( \sum_{i \in N} u^{-1}(\lambda) = W \) if \( W \leq |N|^2 \), and \( F(e) = |N| + u^{-1}(\lambda) \), where \( \lambda > 0 \) is chosen so that \( \sum_{i \in N} u^{-1}(\lambda) = W - |N|^2 \) if \( W > |N|^2 \). Let \( e, e^1 \in \xi, e = (N, u, W), e^1 = (N, u', W) \). Then we can easily show that \( F \) satisfies AG. Now let \( u_1(a) = a^2, u_2(a) = \frac{a^2}{4}, u_3(a) = \frac{a^2}{9}, u_1' = u_1, u_2' = u_2, u_3'(a) = \frac{a^2}{25}, N = \{1, 2, 3\}, W = 33 \). Then for \( a_i = F_i(e) \), \((a_1, a_2, a_3) = (3 + 4, 3 + 8, 3 + 12) = (7, 11, 15)\), \((a_1', a_2', a_3') = (3 + 3, 3 + 6, 3 + 15) = (6, 9, 18)\). Let \( e^2 = (N, u', 41), e^3 = (\{1, 2\}, \{u_1, u_2\}, 18) \) such that \( e^2, e^3 \in \xi\). \((a_1^2, a_2^2, a_3^2) = (3 + 4, 3 + 8, 3 + 20) = (7, 11, 23), (a_1^3, a_2^3, a_3^3) = (2 + \frac{14}{3}, 2 + \frac{28}{3}) = (\frac{20}{3}, \frac{34}{3}) \neq (7, 11)\). From \( e^2, e^3, F \) does not satisfy SL. Therefore, AG does not imply SL.

**Proposition 3.** CY implies SE.

*Proof.* Let \( e = (N, u, W) \in \xi \) and \( e' = (N, u', W') \in \xi \) and let \( M \subseteq N \) and \( M \subseteq N' \) such that \( u_M = u'_M \). If \( \sum_{i \in M} F_i(e) = \sum_{i \in M} F_i(e') \), by definition of CY, \( F_M(e) = F_M(e') \), so that SE holds. \( \square \)

Additionally, it can also be shown (Moreno-Ternero and Roemer, 2005) that RM implies RC.

Now, we show that CY, and RC is a strictly weaker condition than SL by showing that there exists a rule that satisfies CY, and RC but fails to satisfy RM. Note that a rule satisfies SL if and only if it satisfies CY and RM.

**Example 2.** Let \( e = (N, u, W) \) such that \( N = \{1, 2, \ldots, |N|\} \), and \( W > 0 \). We define a rule \( F \) as \( F_1(\lambda) = \lambda + (\lambda - 3)^2 - 1 \) if \( 2 \leq \lambda < 4 \) and \( F_1(\lambda) = \lambda \) otherwise, \( F_2(\lambda) = 2\lambda - (\lambda - 3)^2 + 1 \) if \( 2 \leq \lambda < 4 \) and \( F_2(\lambda) = 2\lambda \) otherwise,
$F_i(\lambda) = i\lambda$ for $i = 3, 4, \ldots$. Then, $F$ satisfy RC since every $F_i$ ($i = 1, 2, 3, \ldots$) is continuous in $\lambda$. Since $\frac{dF_2(\lambda)}{d\lambda} = 8 - 2\lambda > 0$ if $2 \leq \lambda < 4$ and $\frac{dF_2(\lambda)}{d\lambda} = 2 > 0$ otherwise, and $\frac{dF_i(\lambda)}{d\lambda} = i > 0$ for $i = 3, 4, \ldots$, $F_i(\lambda)$ ($i = 2, 3, \ldots$) is strictly increasing. Meanwhile, $\frac{dF_1(\lambda)}{d\lambda} = 2\lambda - 5$ if $2 \leq \lambda < 4$ and $\frac{dF_1(\lambda)}{d\lambda} = 1 > 0$ otherwise. Therefore, $\frac{dF_1(\lambda)}{d\lambda} < 0$ when $2 < \lambda < \frac{5}{2}$ and $\frac{dF_i(\lambda)}{d\lambda} \geq 0$ otherwise.

We can easily show that for any $\lambda > 0$, $F_1(\lambda) < F_2(\lambda) < F_3(\lambda) < \cdots$. Since $F_1(\lambda) + F_2(\lambda) = 3\lambda$ is strictly increasing in $\lambda$ and every $F_i$ except $F_1$ are strictly increasing in $\lambda$, any partial sum of $F_i$s are strictly increasing in $\lambda$.

Therefore, for each given partial wealth, there exists unique $\lambda$ that equals the sum of partial functions and given wealth, and it means that $F$ satisfies CY. However, since $W = \sum_{i \in N} F_i(\lambda)$ is strictly increasing in $\lambda$, by the case that $2 < \lambda < \frac{5}{2}$ with $F_1(\lambda)$, $F$ fails to satisfy RM.

Remark. We can also show that SE, and RC is a strictly weaker condition than AG by the same rule above.

We can arrange the axioms through the preceding discussion as below.
We will show that replacing SL to a strictly weaker condition as CY+RC, or AG is possible, but to SE+RC is impossible, that is a strictly weaker condition than CY+RC, or AG.

4 Characterizations

Let \( \Phi \) be the class of all functions \( \varphi : \mathbb{R}_+^2 \cup (0, 0) \rightarrow \mathbb{R}_+ \), continuous on its domain and nondecreasing, such that \( \inf\{\varphi(x, y)\} = \varphi(0, 0) = 0 \) and, for all \( (x, y) > (z, t), \varphi(x, y) > \varphi(z, t) \). Let \( \varphi \) be a function in the class \( \Phi \). For all \( i \in I \), define the function \( \psi_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) that determines the \( \varphi \)-value that agent \( i \) achieves, depending on the wealth she receives, i.e., \( \psi_i(w) = \varphi(w, u_i(w)) \) for all \( w \in \mathbb{R}_+ \). Then we can define the corresponding index--egalitarian rule as the rule introduced by Moreno-Ternero and Roe-
that equalizes the $\varphi$-value across individuals in an economy.

**Index-Egalitarian Rule** $E^\varphi$: For all $N \in \mathcal{N}$ and all $i \in N$, for $e = (N, u, W)$, $E^\varphi_i(e) = \psi^{-1}_i(\lambda)$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} \psi^{-1}_i(\lambda) = W$.

We introduce another rule similar to *Index-Egalitarian* rule but have a larger domain. *n-Index-Egalitarian* rule is a similar rule to *Index-Egalitarian* rule but allows different $\lambda$-value for different population.

**n-Index-Egalitarian Rule** $E^\varphi_n$: For each group that contains all $N \in \mathcal{N}$ such that $|N| = n$ ($n = 1, 2, \ldots, 3$) and all $i \in N$, and for $e = (N, u, W)$, $\varphi_n \in \Phi$, and $\psi_n(w) = \varphi_n(w, u_i(w))$, $(E^\varphi_n)_i(e) = (\psi_n)_i^{-1}(\lambda_n)$, where $\lambda_n > 0$ is chosen so that $\sum_{i \in N} (\psi_n)_i^{-1}(\lambda_n) = W$.

Note that for all $i \in I$, $\psi_i^{-1}$ is a continuous, strictly increasing, and unbounded function that satisfies $\psi_i^{-1}(0) = 0$.

**Proposition 4.** If $F \in E^\varphi$, then $F \in E^\varphi_n$.

**Proof.** Let $F$ that satisfies $F \in E^\varphi$. Then, by definition of $E^\varphi$, for all the groups that contains all $N \in \mathcal{N}$ such that $|N| = n$ ($n = 1, 2, \ldots, 3$) and all $i \in N$, and for $e = (N, u, W)$, $E^\varphi_i(e) = \psi_i^{-1}(\lambda)$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} \psi_i^{-1}(\lambda) = W$. Let $\psi_1 = \psi_2 = \psi_3 = \cdots \psi$ and $\lambda_1 = \lambda_2 = \lambda_3 = \cdots \lambda$. Then, by definition of $E^\varphi_n$, $F \in E^\varphi_n$. \qed

All the rules within the family $E^\varphi_n(e)$ satisfy priority, and agreement. More remarkably, there is no other rule that satisfies these properties simultaneously. Additionally, all the rules within the family $E^\varphi(e)$ satisfy priority, consistency, and resource continuity. More remarkably, there is no other rule that satisfies these properties simultaneously. Note that by Proposition 4, $E^\varphi_n$ has a larger domain than $E^\varphi$.
Theorem 1. A rule $F$ satisfies priority and agreement if and only if $F \in \{E^\varphi_n\}_{\forall \varphi \in \Phi}$.

Theorem 2. A rule $F$ satisfies priority, consistency, and resource continuity if and only if $F \in \{E^\varphi\}_{\forall \varphi \in \Phi}$.

Corollary. A rule $F$ satisfies priority, separability, and resource monotonicity if and only if $F \in \{E^\varphi_n\}_{\forall \varphi \in \Phi}$.

We can easily prove Corollary by Theorem 1 and Proposition 1. Lemma 1 is shown in Moreno-Ternero and Roemer (2005).

Lemma 1. If $F$ satisfies PR, CY and RC, then $C(F,i,\alpha)$ is downward sloping.

Lemmas 2, 3, and 4 are shown in Moreno-Ternero and Roemer (2006).

Lemma 2. If $F$ satisfies PR and RM, and $C(F,i,\alpha)$ is downward sloping, then for all $\varphi : \mathbb{R}^2_+ \cup \{(0,0)\} \rightarrow \mathbb{R}_+$ such that $\varphi(a,b) = \alpha$, where $\alpha \in \mathbb{R}_+$ is the number for which $(a,b) \in C(F,i,\alpha)$, $\varphi \in \Phi$ and $\varphi$ is continuous on $\mathbb{R}^2_+$.

By using the same process as in Moreno-Ternero and Roemer (2006), $C(F,i,\alpha)$ be substituted for $C_X(F,i,\alpha)$.

Lemma 3. If $F$ satisfies RC, and $C(F,i,\alpha)$ satisfies download sloppiness and covering domain property, then for all $\varphi : \mathbb{R}^2_+ \cup \{(0,0)\} \rightarrow \mathbb{R}_+$ such that $\varphi(a,b) = \alpha$, where $\alpha \in \mathbb{R}_+$ is the number for which $(a,b) \in C(F,i,\alpha)$, $\varphi \in \Phi$ and $\varphi$ is continuous on $\mathbb{R}^2_+$.

Lemma 4. If $F$ satisfies PR and RC, then for any $(i,\alpha) \in I \times \mathbb{R}_+$, there exists $e \in \xi$ such that $F_i(e) = \alpha$.

4.1 Proof of Theorem 1.

It is proven in Moreno-Ternero and Roemer (2006) that all the $E^\varphi$ rules are impartial and satisfy PR and AG. Conversely, let $F$ be an impartial rule that satisfying PR and AG. If the rule $F$ is independent of $u$, all agents are treated
equally by the amount of wealth. By impartiality, for all \( e = (N, u, W) \) and all \( i \in N, F_i(e) = \frac{W_i}{\|N\|} \), which corresponds to the Index-Egalitarian Rule. We will now check the case that \( F \) is dependent on \( u \).

Since that \( F \) satisfies PR and RC, by Lemma 4, \( C_N(F, i, \alpha) \) is not empty.

Now, using Lemma 5, 6, and 7, we show that the family of curves \( \{C_N(F, i, \alpha) : \alpha \in \mathbb{R}_+ \} \) is the isoquant map of an appropriate function \( \varphi \in \Phi \) and that \( F = E^\varphi \).

We show that for any \( N \), any \( C_N(F, i, \alpha) \) is downward sloping.

**Lemma 5.** For all \( N, C_N(F, i, \alpha) \) is downward sloping.

**Proof.** We need to prove that if \((a, b), (a', b') \in C_N(F, i, \alpha) \) and \( a' > a \), then \( b' < b \). Suppose, to the contrary, that \( a' > a \) and \( b' \geq b \). By definition, there exist \( e = (N, u, W), e' = (N, u', W') \in E_N(F, i, \alpha) \), and \( j, k \in N \) such that \((a, b) = (F_j(e), u_j(F_j(e))) \) and \((a', b') = (F_k(e'), u_k(F_k(e'))) \). Let \( M \) be a subset of \( N \) that contains all agents \( i \in N \) such that \( u_i = u_i' \) and \( W^* = W + a' - F_l(e) \) for some \( l \in M \) and \( l \neq i \).

First, we assume that \([\#M] \geq 2 \). Since \( e = (N, u, W), e' = (N, u', W') \in E_N(F, i, \alpha), i \in M \). Meanwhile, for all \( m, u_m \) is continuous, strictly increasing and \( F \) satisfies RC. Therefore we can set \( u^* \) such that \( e^* = (N, u^*, W^*) \) \( u^*_m = u_m \ \forall m \neq l \) and \( u^*_l = \overline{u_l} \). (\( \overline{u_l} \) is an adjusted function from \((N, u, W^*) \) that makes the wealth of agent \( l \) to be \( a' \)). Then, \( \sum_{m \in N \setminus l} F_m(e) = \sum_{m \in N \setminus l} F_m(e^*) = W - F_l(e) \). By SE, \( \forall m \in N, m \neq l F_m(e) = F_m(e^*) \) and \( F_m(e) = F_m(e^*) = \alpha \), which implies that \( e^* \in E_N(F, i, \alpha) \). Without loss of generality, we assume that \( F_l(e) < a' \). To increase the wealth of agent \( l \) from the economy \( e \) to \( e^* \), \( u_l > u_l^* \) must be satisfied. Note that for any agent \( m, u_m > u_m^* \) means that for any wealth \( a, u_m(a) > u_m^*(a) \). Let \( b^* = u_l^*(F_l(e^*)) \). \( (F_j(e^*), u_j(F_j(e^*))) = (a, b), (F_l(e^*), u_l(F_l(e^*))) = (a', b^*) \). Therefore, by PR, \( b' < b \leq b' \) since \( a < a' \). Let \( u_{m, l}^* = u_m^* \) if \( n \neq l \), \( u_{l,l}^* = u_l^*(i.e., \forall m \in M u_{m,l}^* = u_m^*, \forall m \in N \setminus M u_{m,l}^* = u_m^*) \). Let \( W^* \) such that \( e^* = (N, u^{**}, W^{**}) \in E_N(F, i, \alpha) \). For \( m \in N \setminus \{l\} \), \( u_{m,l}^{**} = u_{m,l}^* \) and \( F_i(e^{**}) = F_i(e'), i \in N \setminus \{l\} \), and \( F_{N \setminus \{l\}}(e^{**}) = F_{N \setminus \{l\}}(e') \) (by SE). Therefore (i) \( F_k(e^{**}), u_k^{**}(F_k(e^{**})) = (a', b') \). For \( m \in M \)
Lemma 6. \( C_N(F, i, \alpha) : \alpha \in \mathbb{R}_+ \) is a collection of disjoint sets.

**Proof.** Let \( \alpha_1 > \alpha_2 \). Suppose that \((a, b) \in C_N(F, i, \alpha_1) \cap C_N(F, i, \alpha_2) \). Let \( e_1 = (N, u_1, W_1) \in E_N(F, i, \alpha_1) \) and \( j \in N \) such that \((F_i(e_1), u_j(F_j(e_1))) = (a, b)\). By Lemma 2, there exists \( e_2 = (N, u_1, W_2) \) such that \( F_i(e_2) = \alpha_2 \). By RM, if \( W_1 > W_2 \) then \( F(e_1) > F(e_2) \), and if \( W_1 < W_2 \) then \( F(e_1) < F(e_2) \). Since \( \alpha_1 > \alpha_2 \), \( F_i(e_1) > F_i(e_2) \), and therefore \( W_1 > W_2 \). Then, by RM, \( F_j(e_2) < F_j(e_1) \), \( u_j(F_j(e_2)) < u_j(F_j(e_1)) \). Therefore, \((F_j(e_2), u_j(F_j(e_2))), (F_j(e_1), u_j(F_j(e_1))) \in C_N(F, i, \alpha_2)\) contradicts with the fact that \( C_N(F, i, \alpha_2) \) is downward sloping. \( \Box \)

Next, we show that \( C_N(F, i, \alpha) \) has a covering domain property, that is, if \( \alpha_1 > \alpha_2 \), then \( C_N(F, i, \alpha_1) \) lies above \( C_N(F, i, \alpha_2) \). Similar process to the proof of Lemma 7 is in Moreno-Ternero and Roemer (2006).

**Lemma 7.** If \( \alpha_1 > \alpha_2 \), then \( C_N(F, i, \alpha_1) \) lies above \( C_N(F, i, \alpha_2) \).

**Proof.** We divide the proof into two steps: We show first that \((i) \) for all \((a, b) \in C(F, i, \alpha_2) \) there exists \((a', b') \in C(F, i, \alpha_1) \) such that \((a, b) < (a', b') \),
and then (ii) there is no \((a'', b'') \in C(F, i, \alpha_2)\) and \((a, b) \in C(F, i, \alpha_1)\) such that \((a'', b'') > (a, b)\).

Fix \(\alpha_1 > \alpha_2\) and \(N\) such that \(i, j \in N\). Let \((a, b) \in C_N(F, i, \alpha_2)\). Then there exists \(e = (N, u, W) \in E_N(F, i, \alpha_2)\) such that \((F_j(e), u_j(F_j(e))) = (a, b)\).

Let \(e' = (N, u, W') \in E_N(F, i, \alpha_1)\). Since \(\alpha_2 = F_i(e) < F_i(e') = \alpha_1\), by RM, \(W < W'\). Let \((F_j(e'), u_j(F_j(e'))) = (a', b')\). Then, by RM, \((a, b) < (a', b')\). Therefore, (i) holds.

Now, for \((a, b) \in C_N(F, i, \alpha_1)\), suppose that there exists \((a'', b'') \in C_N(F, i, \alpha_2)\) such that \((a'', b'') > (a, b)\). Then, by (i), there exist \((a^*, b^*) \in C_N(F, i, \alpha_1)\) such that \((a^*, b^*) > (a'', b'')\). Therefore, \((a^*, b^*) > (a, b)\), which contradicts that \(C_N(F, i, \alpha_1)\) is downward sloping.

Let \(\varphi\) be the function defined in Lemma 2. Then, by the fact that \(F\) satisfies PR, RM, and by Lemma 2 and Lemma 5, \(\varphi \in \Phi\) and \(\varphi\) is continuous on \(\mathbb{R}^2_+\). The proof of the theorem concludes by showing that \(F = E^\varphi\), i.e., \(F(N, u, W) = E^\varphi(N, u, W)\) for all \((N, u, W) \in \xi\). We fix \(e = (N, u, W) \in \xi\) and the standard agent \(i\) for \(C_N(F, i, \alpha)\).

If \(i \in N\), for \(\lambda = F_i(e)\), \(\forall j \in N\) \((F_j(e), u_j(F_j(e))) \in C_N(F, i, \lambda)\). By Lemma 1, \(\forall j, \psi_j(F_j(e)) = \varphi(F_j(e), u_j(F_j(e))) = \lambda\). Since \(\sum_{j \in N} F_j(e) = W\), \(F(e) = E^\varphi(e)\) by definition.

We now consider the case when \(i \notin N\). Let \(j, k \in N\) and \(F_j(e) = a_j\), \(F_k(e) = a_k\), \(a_j = u_j(a_j)\), \(b_k = u_k(a_k)\). Then \((a_k, b_k) \in C_N(F, j, a_j)\).

By Lemma 4 and Lemma 5, there exists a unique \(\lambda\) such that \((a_j, b_j) \in C_N(F, i, \lambda)\). Let \(e' = (N, u', W')\) be such that for \(l \in N \setminus \{j, k\}\), \(u'_l = u_l\), \(u'_{N \setminus \{l\}} = u_{N \setminus \{l\}}\) and \(F_i(e') = \lambda\). Since \(u'_i = u_i\), and every agent is characterized only by its individual function, we can consider \(l\) as \(i\). Then, \(F_j(e') = a_j\) since \((a_j, b_j) \in C_N(F, i, \lambda)\), and \(F_k(e') = a_k\) since \((a_k, b_k) \in C_N(F, j, a_j)\). Therefore, \((a_k, b_k) \in C_N(F, i, \lambda)\). This shows that for all \(n \in N, (F_n(e), u_n(F_n(e))) \in C_N(F, i, \lambda)\). By Lemma 1, for all \(n \in N\), \(\psi_n(F_n(e)) = \varphi(F_n(e), u_n(F_n(e))) = \lambda\). Since \(\sum_{n \in N} F_n(e) = W\), \(F(e) = E^\varphi(e)\) by definition.
4.2 Proof of Theorem 2.

\( C(F, i, \alpha) \) is downward sloping by Lemma 1.

**Lemma 8.** \( \{ C(F, i, \alpha) : \alpha \in \mathbb{R}_+ \} \) is a collection of disjoint sets.

**Proof.** Let \( \alpha_1 > \alpha_2 \). Suppose that \((a, b) \in C(F, i, \alpha_1) \cap C(F, i, \alpha_2) \). Let \( e_1 = (N_1, u_1, W_1) \in E(F, i, \alpha_1) \), \( e_2 = (N_2, u_2, W_2) \in E(F, i, \alpha_2) \) and \( j \in N_1 \), \( k \in N_2 \) such that \((a, b) = (F_j(e_1), u_j(F_j(e_1))) = (F_k(e_2), u_k(F_k(e_2))) \). Let \( e' = (\{i, j, k\}, (u_i, u_j, u_k), W') \in E(F, i, \alpha_1) \) and \( e'' = (\{i, j, k\}, (u_i, u_j, u_k), W'') \in E(F, i, \alpha_2) \). If \( F_j(e') > a \) then \( (F_j(e'), u_j(F_j(e'))) > (a, b) \), and if \( F_j(e') < a \) then \( (F_j(e'), u_j(F_j(e'))) < (a, b) \). Since \((a, b) \in C(F, i, \alpha_1) \) and \((F_j(e'), u_j(F_j(e'))) \in C(F, i, \alpha_1) \), both \( F_j(e') > a \) and \( F_j(e') < a \) contradict the fact that \( C(F, i, \alpha_1) \) is downward sloping. Therefore \( F_j(e') = a \), and since \( u_j(a) = u_k(a) = b \), by the same process, \( F_k(e') = a \). From \((a, b) \in C(F, i, \alpha_2) \), by the same process above, \( F_j(e'') = F_k(e'') = a \). From \((F_j(e''), u_j(F_j(e''))) = (F_j(e''), u_j(F_j(e''))) = (a, b), (\alpha_1, u_i(\alpha_1)) \in C(F, j, a) \) and \((\alpha_2, u_i(\alpha_2)) \in C(F, j, a) \). Since \( \alpha_1 > \alpha_2, (\alpha_1, u_i(\alpha_1)) > (\alpha_2, u_i(\alpha_2)) \), which contradicts the fact that \( C(F, j, a) \) is downward sloping. \( \square \)

**Lemma 9.** If \( \alpha_1 > \alpha_2 \), then \( C(F, i, \alpha_1) \) lies above \( C(F, i, \alpha_2) \).

**Proof.** We first show that if \( \alpha_1 > \alpha_3 \), then \( C(F, i, \alpha_1) \) lies above \( C(F, i, \alpha_3) \) or lies below \( C(F, i, \alpha_3) \). Note that \( C(F, i, \alpha_1) \) lies below \( C(F, i, \alpha_3) \) means that \( C(F, i, \alpha_3) \) lies above \( C(F, i, \alpha_1) \). Let \( \alpha_1 > \alpha_3 \). We will show that for any \( \alpha_2 \) such that \( \alpha_1 > \alpha_2 > \alpha_3 \), \( C(F, i, \alpha_2) \) lies between \( C(F, i, \alpha_1) \) and \( C(F, i, \alpha_3) \). Without loss of generality, assume that \( C(f, i, \alpha_1) \) lies below \( C(F, i, \alpha_3) \). Suppose that there exists \( \alpha_2 \) such that \( \alpha_1 > \alpha_2 > \alpha_3 \) and \( C(F, i, \alpha_2) \) lies below \( C(F, i, \alpha_1) \). Then, there exists \( e_1 = (\{i, j\}, (u_i, u_j), W_1) \in E(F, i, \alpha_1), e_2 = (\{i, j\}, (u_i, u_j), W_2) \in E(F, i, \alpha_2), e_3 = (\{i, j\}, (u_i, u_j), W_3) \in E(F, i, \alpha_3) \) such that \( (F_j(e_2), u_j(F_j(e_2))) < (F_j(e_1), u_j(F_j(e_1))) < (F_j(e_3), u_j(F_j(e_3))) \). By
RC, $C(F, i, \alpha_2)$ moves continuously to $C(F, i, \alpha_3)$ as $\alpha_2$ changes continuously to $\alpha_3$. That is, there exists $\alpha_4$ such that $\alpha_2 > \alpha_4 > \alpha_3$ and $C(F, i, \alpha_4)$ coincides with $C(F, i, \alpha_1)$, which contradicts the fact that $\{C(F, i, \alpha) : \alpha \in \mathbb{R}_+\}$ is a collection of disjoint sets. We can apply the same process above when $C(F, i, \alpha_1)$ lies above $C(F, i, \alpha_3)$. Therefore, $C(F, i, \alpha)$ is monotonic in $\alpha$.

Now, we show that $C(F, i, \alpha_1)$ lies above $C(F, i, \alpha_3)$. Assume that $C(F, i, \alpha_1)$ lies below $C(F, i, \alpha_3)$. Let $e = \{(i, j), (u_i, u_j), W\}$. Since $C(F, i, \alpha_1)$ lies below $C(F, i, \alpha_3)$ for any $\alpha_1 > \alpha_3$, $F_j(e)$ decreases as $F_i(e)$ increases. Let $\beta > 0$ be the value such that $u_i(\beta) = u_j(F_j(e))$. Let $\beta_1, \beta_2$ be any value such that $\beta_1 > \beta > \beta_2 > 0$. By Lemma 3, there exists $e' = \{(i, j), (u_i, u_j), W'\} \in E(F, i, \beta_1)$, $e'' = \{(i, j), (u_i, u_j), W''\} \in E(F, i, \beta_2)$. (i) If $F_i(e) > F_j(e)$ then $u_i(F_i(e)) < u_j(F_j(e)) = u_i(\beta)$ (by PR), and $F_i(e) < \beta$ (since $u_i$ is strictly increasing). Then, $F_i(e') = \beta_1 > \beta > F_i(e)$, therefore $F_j(e') < F_j(e)$. Meanwhile, $u_i(F_i(e')) > u_i(\beta) = u_j(F_j(e)) > u_j(F_j(e'))$. Since $F_i(e') > F_i(e) > F_j(e) > F_j(e')$, $(F_i(e'), u_i(F_i(e'))) > (F_j(e'), u_j(F_j(e'))))$, which contradicts PR. (ii) If $F_i(e) < F_j(e)$ then $u_i(F_i(e)) > u_j(F_j(e))$(by PR), and $F_i(e) > \beta$. Then, $F_i(e'') = \beta_2 < \beta < F_i(e)$, therefore $F_j(e'') > F_j(e)$. Meanwhile, $u_i(F_i(e'')) < u_i(\beta) = u_j(F_j(e)) < u_j(F_j(e'))$. Since $F_i(e'') < F_i(e) < F_j(e) < F_j(e'')$, $(F_i(e''), u_i(F_i(e''))) < (F_j(e''), u_j(F_j(e'')))$, which contradicts PR. Therefore, from (i) and (ii), $C(F, i, \alpha_1)$ lies above $C(F, i, \alpha_3)$ for any $\alpha_1 > \alpha_3$.

Let $\varphi$ be the function defined in Lemma 3. Then, by the fact that $F$ satisfies PR, RC, and by Lemma 1, Lemma 2, and Lemma 5, $\varphi \in \Phi$ and $\varphi$ is continuous on $\mathbb{R}_+^2$. The proof of the theorem concludes by showing that $F = E^\varphi$, i.e., $F(N, u, W) = E^\varphi(N, u, W)$ for all $(N, u, W) \in \xi$. We fix $e = (N, u, W) \in \xi$ and the standard agent $i$ for $C(F, i, \alpha)$.

If $i \in N$, for $\lambda = F_i(e)$, $\forall j \in N \ (F_j(e), u_j(F_j(e))) \in C(F, i, \lambda)$. By Lemma 1, $\forall j, \psi_j(F_j(e)) = \varphi(F_j(e), u_j(F_j(e))) = \lambda$. Since $\sum_{j \in N} F_j(e) = W$, $F(e) = E^\varphi(e)$ by definition.

We now consider the case when $i \notin N$. Let $j, k \in N$ and $e = (N, u, W)$. 18
Let $w_j = F_j(e)$, $w_k = F_k(e)$. Then $(w_k, u_k(w_k)) \in C(F, j, w_j)$, $(w_j, u_j(w_j)) \in C(F, k, w_k)$. By Lemma 3, there exists $e_1 = (\{i, j, k\}, (u_i, u_j, u_k), W_1)$ such that $F_j(e_1) = w_j$, then $(F_k(e_1), u_k(F_k(e_1))) \in C(F, j, w_j)$. If $F_k(e_1) > w_k$ then $(F_k(e_1), u_k(F_k(e_1))) > (w_k, u_k(w_k))$, which contradicts the fact that $C(F, j, w_j)$ is downward sloping, and if $F_k(e_1) < w_k$ then $(F_k(e_1), u_k(F_k(e_1))) < (w_k, u_k(w_k))$, which contradicts the fact that $C(F, j, w_j)$ is downward sloping. Therefore $F_k(e_1) = w_k$. By the same process above, $F_N(e_1) = F(e)$. Let $\lambda = F_i(e_1)$. Then $(F_i(e_1), u_l(F_i(e_1))) \in C(F, i, \lambda)$ for all $l \in N$. Thus, $\psi_l(F_i(e_1)) = \lambda$ for all $l \in N$. Since $\sum_{l \in N} F_i(e_1) = \sum_{l \in N} F_l(e) = W$, it follows that $F(e) = E^\pi(e)$.

**References**


Moreno-Ternero, J.D., and J.E. Roemer (2006), A common ground for resource and welfare egalitarianism


5 Appendix.

Next, we show that it is impossible to replace AG to SE in Theorem 1 by the following example that satisfies PR, and SE. Note that SE and RM together implies AG.

**Example 3.** Let $e = \{1, 2, 3\}, u, W \in \xi$ such that $u(w) = (w^2, \frac{w^2}{5}, \frac{w^2}{27})$. Let $(\hat{F}_1(e), \hat{F}_2(e), \hat{F}_3(e)) = (\frac{1}{\sqrt{u_1(1)}}a, 2a, \frac{1}{\sqrt{u_2(1)}}a)$ for $a$ be a constant such that $\sum_N \hat{F}_i(e) = W$. Then, let $F_i(e) = \alpha_i\hat{F}_i(e)$ such that $\alpha_1 = 1 - \frac{3\sqrt{u_1(1)}}{5}$ if $\hat{F}_i(e)\sqrt{u_1(1)} > 5$ and $\alpha_1 = 1$ otherwise, $\alpha_2 = \frac{11}{10}$ if $\alpha_1 = 1 - \frac{3\sqrt{u_1(1)}}{5}$ and $\alpha_2 = 1$ otherwise, $\alpha_3 = 1$ always. If $W \leq 30$, from $\hat{F}_1(e) : \hat{F}_2(e) : \hat{F}_3(e) = 1 : 2 : 3$, $\hat{F}_1(e)\sqrt{u_1(1)} \leq 5$ and therefore $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $F_1(e) + F_2(e) \leq 15$. However, if $W > 30$, then $\hat{F}_1(e) > 5$, $\alpha_1 = \frac{4}{5}$, $\alpha_2 = \frac{11}{10}$, and therefore $F_1(e) : F_2(e) : F_3(e) = 4 : 11 : 15$, $F_1(e) + F_2(e) > 15$. That is, regardless of $u_3$, if $\hat{F}_i(e) \leq 5$, $F_1(e) + F_2(e) \leq 15$ and $F_1(e) : F_2(e) = 1 : 2$, otherwise $F_1(e) + F_2(e) > 15$ and $F_1(e) : F_2(e) = 4 : 11$. Next, we consider when $u_1$ changes such that $u_1(w) = b^3w^2$ $(0 < b < 5)$. Then, $\alpha_1 = 1 - \frac{b}{5}$, $\alpha_2 = \frac{11}{10}$, and $F_1(e) : F_2(e) : F_3(e) = \frac{5}{6} - 1 : 11 : 15$ if and only if $a \leq 5$. That is, if $a > 5$ $F_2(e) + F_3(e) > 26$ and $F_2(e) : F_3(e) = 11 : 15$, and if $a \leq 5$ $F_2(e) + F_3(e) \leq 25$ and $F_2(e) : F_3(e) = 2 : 3$. Trivially, by the feature of $u$, $F_N(e)$ doesn’t change when $u_2$ changes. Therefore, SE holds. Additionally, we can easily show that $F$ satisfies PR. Now, let’s consider the two economy $e^1 = (F, u, 30)$ and $e^2 = (F, u, 36)$. From $F(e^1) = (5, 10, 15)$, $F(e^2) = (4.8, 13.2, 18)$, $(F_1(e^1), u_1(F_1(e^1))) = (5, 25)$, $(F_2(e^1), u_2(F_2(e^1))) = (10, 12.5)$, $(F_1(e^2), u_1(F_1(e^2))) = (4.8, 23.04)$, $(F_2(e^2), u_2(F_2(e^2))) = (13.2, 21.78)$. Meanwhile, for any $\hat{\varphi} \in \Phi$, $\hat{\varphi}(5, 25) > \hat{\varphi}(4.8, 23.04)$ and $\hat{\varphi}(10, 12.5) < \hat{\varphi}(13.2, 21.78)$ which implies that $F \notin E^\varphi$ and $F \notin E_{n}^{\varphi}$. Consequently, $F$ satisfies priority, and separability, but fails to satisfy $F \in E^\varphi$ and $F \in E_{n}^{\varphi}$. 