A NETWORK CENTRALITY APPROACH TO
COALITIONAL STABILITY

MATT ELLIOTT∗ AND BENJAMIN GOLUB**

ABSTRACT. We study games in which each player simultaneously exerts costly effort that provides different benefits to some of the other players. The static analysis of the game yields a prediction no cooperation, while a standard repeated games approach yields a folk theorem. To obtain more refined predictions about repeated play, we start from the observation that outcomes in such settings are typically negotiated in multilateral meetings involving various subsets of agents. Thus, our goal is to find and describe effort profiles that can be sustained in equilibrium despite the possibility of coordinated coalitional deviations.

In general, even the existence of such equilibria is a difficult matter. This paper argues that there are simple, robust solutions that can be found by analyzing the setting as a network of benefit flows among agents. In particular, we discuss connections among three concepts: coalition-proof equilibria; effort profiles satisfying a certain network centrality condition; and Lindahl equilibria (which are "Walrasian" solutions in a public goods environment). We use these connections to find certain efficient coalition-proof outcomes exist, in which each agent’s effort is equal to a sum (appropriately weighted) of efforts of those whose contributions help him. The results are obtained without parametric assumptions, using the theory of general equilibrium and its relation to the core, along with the Perron-Frobenius spectral theory of nonnegative matrices.

Keywords: centrality, repeated games, strong Nash equilibrium, coalitional deviations, core, β-core, competitive equilibrium, general equilibrium, Perron-Frobenius, spectral theory.

Date Printed. June 3, 2012.

*Microsoft Research.
Email: matt.elliott@stanford.edu, web: http://www.stanford.edu/~mle1.

**Department of Economics, MIT.
Email: bgolub@mit.edu, web: http://www.stanford.edu/~bgolub.

We thank Nageeb Ali, Kyle Bagwell, Arun Chaudrsekh, Glenn Ellison, Matt Jackson, Anil Jain, Alex Frankel, Hari Govindan, Bengt Holmstrom, Matthew O. Jackson, Vikram Manjunath, Muriel Niederle, Phil Reny, Andy Skrzypacz, Moshe Temenholtz, Juuso Toikka, Bob Wilson, and Muhamet Yildiz for their comments and suggestions. Elliott gratefully acknowledges financial support from the Davidow fellowship administered through SIEPR. Golub gratefully acknowledges financial support from the Lieberman fellowship at Stanford and a Prize Fellowship in Economics, History, and Politics at Harvard.
1. Introduction

Modeling negotiations among many countries to solve a collective action problem, such as limiting greenhouse gas emissions, is difficult. First, approaching the problem in general requires dealing with the heterogeneity inherent in the setting: each country, by limiting its emissions, may confer different benefits on each other country, while also benefiting differently from other countries’ emissions reductions. Second, the multilateral nature of negotiations and the extensive communication that takes place means that a stable outcome should be robust to coordinated coalitional deviations. Indeed, coalitional stability alone makes the problem hard. Aumann (1959), Bernheim, Peleg and Whinston (1987), and others have demonstrated that, in repeated games, one cannot always guarantee the existence of an equilibrium when coalitional deviations are possible. Fortunately, collective action problems of interest such as climate negotiations have some additional structure that mitigates this potential problem. When a country imposes a more stringent limit on its greenhouse gas emissions than its unilateral optimum, it incurs costs but benefits the other countries (possibly to different degrees). We model this: i.e., we consider a stage game in which agents’ payoffs are decreasing in their own actions (in this case, emissions reductions) but weakly increasing in other agents’ actions. In the repeated play of such a stage game with patient players, we show not only that equilibria robust to coalitional deviations exist, but also that a class of them can be characterized by a simple network centrality condition. We do so while allowing for heterogeneities in the benefit flows different agents can confer on each other, without any functional form restrictions.

A brute force verification than an equilibrium outcome is robust to coalitional deviations requires considering $2^n - 1$ possible deviating coalitions, and many possible deviations for each coalition.\(^1\) However, viewing the problem in terms of a network of benefit flows (related to the Jacobian matrix of the game) helps us identify a single condition that, if satisfied, ensures that an outcome is coalitionally stable. Moreover, there always exists a coalitionally stable outcome satisfying our condition. The network we use is constructed based on the heterogeneous marginal benefits yielded by different agents’ action choices: the weight $G_{ij}$ of a link to $i$ from $j$ captures the marginal benefit $i$ receives from $j$ per marginal unit of $j$’s cost. The matrix of weights $G$, which we refer to as the gift matrix, is the key object we analyze to find efficient and equilibrium outcomes of the game.

Our first result shows that actions are on the Pareto frontier if and only if the gift matrix $G$ has a largest eigenvalue (spectral radius) of exactly 1. There is a natural intuition for this result. Suppose an agent increases her contribution slightly, and all agents who benefit “pass forward” all the resulting benefits by increasing their actions until they are just as well off as before the increase – and so on, with only the original contributor keeping the benefits she gets back. If more benefits are eventually returned to the first agent than her up-front investment of effort, then a Pareto improvement has been constructed. This occurs when the network of benefit flows is exploding at the margin – when its largest eigenvalue is greater than 1. Efficient outcomes are obtained by precisely exhausting all such money pumps.

Another, stronger spectral condition on an action profile – namely, being centrality-stable – ensures that the profile is not only efficient, but coalitionally stable. This is our main result. Define an agent $i$’s contribution $c_i$ at an effort profile as her action times her marginal cost. An effort profile is centrality-stable if the vector of contributions is a right

\(^1\)Deng and Papadimitriou (1994) find a related problem to be computationally intractable.
eigenvector, with corresponding eigenvalue 1, of the matrix $G$. This means that $c_i$, the contribution of $i$, is equal to the sum of terms $G_{ij} \cdot c_j$ as $j$ ranges over the other agents. A typical term in the sum is $j$’s contribution, weighted by how much $j$ helps $i$. Thus, agents who contribute highly are those who have strong links in the gift matrix from others who contribute highly. If this condition is satisfied, then repeated play of the effort profile occurs in some strong Nash equilibrium of the repeated game. This theorem connects—to our knowledge, for the first time—the study of outcomes that are robust to coalitional deviations with the study of network centrality.

To gain intuition for our main result, consider an artificial economy in which each agent can pay every other agent for effort put forth, and agents supply and demand their efforts in a competitive market. While this situation is very different from the non-transferrable-utility game of interest, it turns out (as we explain below) that considering this economy allows us to find and characterize certain coalitionally stable outcomes. In the context of this economy, $G_{ij} \cdot c_j$ can be thought of as the amount $i$ pays to $j$ for the benefits $j$ confers on $i$, and $c_j$ as agent $i$’s income. An action profile being centrality-stable then encodes a necessary condition for a market equilibrium of the artificial economy: budget-balance. The key to our result is that this single condition also implies the existence of prices that clear the market, and thus yields a market equilibrium. Those prices come from a novel application of the Perron-Frobenius Theorem on nonnegative matrices. Lastly, once a market equilibrium is found, it can be used to construct a strong Nash equilibrium of the repeated game of interest—an equilibrium in which all coalitions are playing best responses. The key ideas here are that market equilibria are in the core (appropriately defined) of the stage game, just as in general equilibrium theory; then, adapting ideas of Aumann (1959), one can construct an equilibrium of the repeated game implementing the core outcome.

In Section 5, we extend our analysis to a particular structure of private information. Agents can verify marginal costs and benefits to various actions, but may remain ignorant about other agents’ utility functions away from the margin. Centrality-stable effort profiles are the only ones robust to the misreporting of utilities in this setting, assuming these utility functions are concave.

An alternative definition of centrality-stable profiles is that they are the ones satisfying scaling-indifference: they are characterized by everyone being indifferent to scaling all actions by the same proportionality constant near 1. This can be used to compute centrality-stable points (see Sections 6.3 and 6.4) and to reach these outcomes along gradual paths with the property that at every stage, everyone is willing to take the next step (see Section 6.5). We also discuss the question of whether it might be possible to characterize all sustainable action-profiles using our techniques (Section 6.2).

While we have motivated our analysis by considering the problem of climate change negotiations, there are many other collective action problems that our models fits. When countries negotiate reductions in their tariffs, communication is extensive and encouraged, while tariff reductions provide heterogenous benefits to other countries and are individually costly.\(^\text{2}\) When team members collaborate on a project, coalitional deviations are natural and the effort of one team member provides heterogenous benefits to the other team members.

\(^{2}\)The institutional features of the WTO are crucial for ensuring that tariff reductions by one country do not hurt any other country (see Elliott and Golub, 2011).
A recent literature on networks has related Nash equilibria in one-shot games to spectral and centrality conditions when best responses are linear in others’ actions; key papers in this literature include Balaster, Calvó-Amengol, and Zenou (2006) and Bramoullé, Kranton, and d’Amours (2011). By studying a different problem with group deviations, we have a new structure to exploit, and find different spectral conditions characterizing robust outcomes of repeated, rather than one-shot, interactions. Furthermore, to obtain the spectral conditions in this different setting, we need only concave utility functions instead of specific functional forms that make best responses linear.

2. The Model

2.1. The Game. We begin by defining a stage game $\Gamma$, in which each member of a set $N = \{1, 2, \ldots, n\}$ of players simultaneously chooses an effort level, or action $a_i \in \mathbb{R}_{\geq 0}$. Each player has a utility function $u_i : \mathbb{R}^n_{\geq 0} \rightarrow \mathbb{R}$, and player $i$’s stage game payoff is $u_i(a)$. As a normalization, we assume that $u(0) = 0$. The repeated game $\Gamma^*(\delta)$ is one in which each agent takes a possibly random action $a_{t,i}$ in each of infinitely many discrete periods $t$, and payoffs are given by $U_i = \sum_{t=0}^{\infty} \delta^t u_i(a_t)$. The game is one of complete and symmetric information.

2.2. Equilibrium Concept and Sustainable Actions. Our focus will be on outcomes of the repeated game that are robust to the possibility of coalitional deviations. To make this formal, we first define the standard notion of a strong Nash equilibrium (Aumann, 1959).

Definition. A strong Nash equilibrium of a game $G$ played by the set of players $N$ is a strategy profile $\sigma$ of the game $G$ such that there is no nonempty coalition $M \subseteq N$ and no other strategy profile $\sigma'$ of this game so that:

(i) $\sigma'_i = \sigma_i$ for all $i \notin M$;
(ii) each $i \in M$ strictly prefers $\sigma'$ to $\sigma$.

The key object of study in this paper is the set of sustainable action vectors:

Definition. An action vector $a$ of the stage game $\Gamma$ is sustainable if there is a $\delta < 1$ so that if $\delta \geq \delta$, there is a strong Nash equilibrium $\sigma$ of the repeated game $\Gamma^*(\delta)$, which is also a subgame-perfect Nash equilibrium, in which the infinite repetition of $a$ occurs on the path of play.

The definition requires that repeated play of $a$ be sustained in an equilibrium which deters (with threats of punishments) any coalition from jointly deviating from it. The notion of strong Nash equilibrium places no requirements on play off the equilibrium path, so to rule out equilibria supported by non-credible threats, we also require that the equilibrium is a subgame-perfect Nash equilibrium. Off the equilibrium path, this requires that play be immune to unilateral (but not necessarily coalitional) deviations.

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3We use $\mathbb{R}_{\geq 0}$ (respectively, $\mathbb{R}_{> 0}$) to denote the set of nonnegative (respectively, positive) real numbers. We write $\mathbb{R}^n_{\geq 0}$ (respectively, $\mathbb{R}^n_{> 0}$) for the set of vectors $v$ with $n$ entries so that each entry $v_i$ is in $\mathbb{R}_{\geq 0}$ (respectively, $\mathbb{R}_{> 0}$). When we write an inequality between vectors, e.g. $v > w$, that means the inequality holds coordinate by coordinate, i.e. $v_i > w_i$ for each $i \in N$.

4We sometimes refer to vectors of actions and utilities, so that in the stage game actions $a$ yield payoffs $u(a)$.

5We relax this assumption in Section 5.
One interpretation of sustainable outcomes that can be applied to our collective action problems is that, following a deviation, communication breaks down and only unilateral deviations are then possible. Nevertheless, in Section 6.1 we discuss how our results can be extended to permit a stronger notion of subgame-perfection in which coalitional deviations are possible off the equilibrium path.

One possible concern with considering outcomes derived from strong Nash equilibria is that too many outcomes may be excluded. For example, an outcome can be ruled out because there is a profitable coalitional deviation even though that coalitional deviation is not stable—in the sense that it is subject to a further profitable deviation by a subcoalition. This concern led Berheim, Peleg, and Whinston (1987) to introduce the concept of coalition-proof Nash equilibria. Although sustainable outcomes may exclude too many outcomes, all sustainable outcomes are coalition-proof.

2.3. Assumptions on Utilities. In general, sustainable action vectors do not always exist (Aumann, 1959). To make the study of these objects tractable, some assumptions are necessary. We make these assumptions motivated by the collective action problems we are studying.

Each \( u_i : \mathbb{R}^n_{\geq 0} \rightarrow \mathbb{R} \) is assumed to be a continuously differentiable function, bounded above. We make six further assumptions on the stage game payoffs \( u_i \):

**Assumption 1** (Costly Actions). For every \( i \in N \), agent \( i \) finds it costly, at the margin, to invest effort: \( \frac{\partial u_i}{\partial a_i} < 0 \).

**Assumption 2** (Positive Externalities). Increasing any player’s action level weakly benefits all other players: \( \frac{\partial u_i}{\partial a_j}(a') \geq 0 \) for all \( j \neq i \).

**Assumption 3** (Strictly* Concave Payoffs). Suppose (i) \( a'' = \lambda a + (1 - \lambda)a' \) for \( a \neq a' \) and \( \lambda \in (0, 1) \); and (ii) \( i \) and \( j \) are such that \( a_j \neq a'_j \) and \( \frac{\partial u_i}{\partial a_j}(a'') > 0 \). Then \( u_i(a'') > \lambda u_i(a) + (1 - \lambda)u_i(a') \).

To state our next assumptions, some definitions are useful. An action profile \( a' \in \mathbb{R}^n_{\geq 0} \) Pareto-dominates another profile \( a \in \mathbb{R}^n_{\geq 0} \) if \( u_i(a') \geq u_i(a) \) for all \( i \in N \), and the inequality is strict for some \( i \). We say \( a' \) strictly Pareto-dominates \( a \) if \( u_i(a') > u_i(a) \) for all \( i \in N \). Finally, \( a \) is Pareto-efficient if no other action profile Pareto-dominates it.

**Assumption 4** (Zero is Inefficient). The action \( a = 0 \) is not Pareto-efficient.

**Assumption 5** (Bounded Improvements). There is no \( a \in \mathbb{R}^n_{\geq 0} \) so that for all \( s, t \in \mathbb{R} \) such that \( s > t \geq 0 \), the action profile \( sa \) strictly Pareto-dominates \( ta \).

**Assumption 6** (Connectedness of Benefit Flows). If \( a \in \mathbb{R}^n_{\geq 0} \) and \( M \) is a nonempty proper subset of \( N \), then there exist \( i \in M \) and \( j \notin M \) so that \( \frac{\partial u_i}{\partial a_j}(a) > 0 \).

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6Lemma 2 in the Appendix shows that our assumption implies concavity of the \( u_i \). This strict* concavity assumption is weaker than the usual strict concavity assumption (in which (ii) would be dropped from the definition). Loosely speaking, we require agent \( i \)'s payoff to be strictly concave in agent \( j \)'s action only when \( i \)'s marginal payoff from \( j \) increasing her action is positive. This allows, in particular, for the marginal effect of \( j \) increasing his action on \( i \) to be exactly zero, which would not be possible if we assumed truly strict concavity.
The positive externalities assumption seems to us the most restrictive. The content of this assumption is that there is one direction in which each player can change her action which is always good for others and costly for herself. This direction is independent of the current action levels being played, and is labeled the positive direction.\footnote{There is a broad class of economic applications for which this restriction is reasonable. When team members increase their effort levels, they provide (different) benefits to the other team members; when countries agree to limit their greenhouse gas emissions, they typically provide (different) benefits to other countries. The World Trade Organization (WTO) provides the institutional setting for multilateral tariff reduction negotiations. In a companion paper (Elliott and Golub, 2011) we show how multiple institutional features of the WTO combine to ensure that any country’s tariff reductions can only benefit other countries.}

We do not view the assumption on the connectedness of benefit flows as restrictive. The purpose of the assumption is to ensure that a key matrix in our analysis is irreducible. Were this matrix to be reducible at equilibrium action levels, the statements of our results would change only slightly: we would need to state conditions on each irreducible block of the matrix.

2.4. Key Notions. For our main results, some additional definitions are useful. We define the Jacobian $J(a)$ of the game $\Gamma$ at action levels $a$ to be the $n$-by-$n$ matrix whose $(i,j)$ element is

$$J_{ij}(a) = \frac{\partial u_i}{\partial a_j}(a).$$

Our main result can be stated using only this matrix (see Section 4.4). However, the proofs and certain intuitive interpretations are aided by two additional notions. The first of these is the gift matrix, a close relative of the Jacobian. The second is an alternative way of measuring actions.

The gift matrix $G$ is defined as follows:

$$G_{ij}(a) = \begin{cases} 
J_{ij}(a) - J_{jj}(a) & \text{if } i \neq j \\
0 & \text{otherwise.}
\end{cases}$$

The quantity $G_{ij}$ captures the marginal benefit $j$ provides to $i$ per marginal unit of $j$’s cost.\footnote{One might be concerned that the definition of this matrix is affected by changes such as multiplying one agent’s utility by a scalar. Reassuringly, this type of change does not change any results. On this point, see Remark 1 in Section 3 and Remark 2 in Section 4.}

Finally, we define the contribution of agent $i$ at an action profile $a$ as

$$c_i(a) = -a_i \cdot \frac{\partial u_i}{\partial a_i}(a).$$

One interpretation of this is as follows. Suppose $i$ subcontracted taking the action $a_i$ to an identical copy of herself. If $i$ paid her twin a wage in utiles equal to the twin’s marginal cost, then $i$’s total cost would be $c_i(a)$. Note that by concavity of the $u_i$, the function $c_i$ is increasing in $a_i$ (holding the other components of $a$ fixed).

Finally, we define the centrality-stable action vectors.

Definition. An action vector $a \in \mathbb{R}^n_{\geq 0}$ is centrality-stable if $a \neq 0$ and $G(a)c(a) = c(a)$. In Section 4, we discuss the interpretation of this condition.
3. Pareto Frontier

**Theorem 1.** Under the assumptions in Section 2.3, an interior action profile \( \mathbf{a} \in \mathbb{R}^n_{>0} \) is Pareto-efficient if and only if 1 is a largest eigenvalue of \( \mathbf{G}(\mathbf{a}) \).

An intuitive reason for this result, based on considering “passing forward” benefits, was described in the introduction.

**Remark 1.** The condition that the largest eigenvalue of \( \mathbf{G}(\mathbf{a}) \) is 1 is independent of how different agents’ cardinal utilities are measured. Formally, suppose we define, for each \( i \in N \), new utility functions \( \hat{u}_i(\mathbf{a}) = f_i(u_i(\mathbf{a})) \) for some smooth, strictly increasing functions \( f_i \). If we let \( \hat{\mathbf{G}} \) be the gift matrix obtained from these new utility functions, then 1 is a largest eigenvalue of \( \mathbf{G}(\mathbf{a}) \) if and only if it is a largest eigenvalue of \( \mathbf{G}(\mathbf{a}) \), though the corresponding eigenvectors may differ. The intuition for this is simple. Recall that, in a concave problem, the Pareto-efficient points are exactly those that maximize some weighted sum of agents’ utilities. If \( \mathbf{a}^\star \) maximizes \( \sum_{i \in N} \theta_i u_i \) then to recover the same maximizer under utilities \( \hat{u}_i \), we simply need to adjust the Pareto weights \( \theta \).

Part of the statement of Theorem 1 simplifies when each agent incurs a cost of \( a_i \) for choosing action level \( a_i \). Let \( \mathbf{I} \) be the identity matrix.

**Corollary 1.** Under the assumptions in Section 2.3, suppose \( J_{ii}(\mathbf{a}) = -1 \) for all \( i \) and \( \mathbf{a} \in \mathbb{R}^n_{>0} \). Then an interior action profile \( \mathbf{a} \in \mathbb{R}^n_{>0} \) is Pareto-efficient outcome if and only if 1 is a largest eigenvalue of \( \mathbf{J}(\mathbf{a}) + \mathbf{I} \).

4. Centrality-Stable Actions

4.1. The Main Result. Having characterized the efficient interior action profiles, we can now proceed with the study of ones that are sustainable even under the possibility of coalitional deviations, and, in particular, centrality-stable outcomes. The main theorem is as follows.

**Theorem 2.** Under the assumptions in Section 2.3, the following statements hold:

(i) If \( \mathbf{a} \in \mathbb{R}^n_{>0} \) is centrality-stable, then \( \mathbf{a} \) is sustainable.

(ii) There exists a centrality-stable \( \mathbf{a} \in \mathbb{R}^n_{>0} \).

(iii) If \( \mathbf{a} \in \mathbb{R}^n_{>0} \) is sustainable, then \( \mathbf{a} \) is Pareto-efficient.

Part (i) of Theorem 2 identifies a sufficient condition for a sustainable outcome and part (ii) shows that this sufficient condition is not vacuous. This sufficient condition requires that the contributions be a right eigenvector of the gift matrix with an associated eigenvalue of 1. From Theorem 1 we therefore know that this sustainable outcome is on the Pareto frontier.\(^9\) Part (iii) of Theorem 2 asserts that all sustainable outcomes are on the Pareto frontier. This last part is not especially surprising, as at every point not on the Pareto frontier there is profitable deviation for the grand coalition.

\(^9\)To guarantee that the action profile is interior, note that all actions are strictly positive in all centrality stable action vectors. By definition at least one agent’s contribution is positive, and this requires at least one action level to be positive. Then, by the irreducibility of \( \mathbf{G}(\mathbf{a}) \), the condition \( \mathbf{G}(\mathbf{a})c(\mathbf{a}) = c(\mathbf{a}) \) ensures that all contributions and this all actions are strictly positive.
4.2. **Interpretation and an Example.** To gain some intuition for what it means for an action profile to be centrality-stable, note that the condition requires, for each $i \in N$:

$$c_i = \sum_{j \neq i} G_{ij} c_j. \quad (1)$$

Equation (1) shows that at a centrality-stable profile, each agent's contribution is a weighted sum of the other agents' contributions, where the weight on $c_j$ is proportional to the marginal benefits that $j$ provides to $i$. We interpret this as requiring a form of "fairness". The following example elaborates on this interpretation.

**Example 1.** Suppose each agent benefits from every agent's action (including her own) in the same way, and this common utility depends only on the sum of effort levels:

$$u_i(a) = \nu \left( \sum_{j \in N} a_j \right) - k_i a_i,$$

where $\nu$ is a continuously differentiable, increasing, strictly concave, and bounded function and $k_i \in (0, \nu'(0))$ is a cost parameter for each agent. Note that agents differ only in their marginal costs. Applying (1) to this special case (which satisfies all the assumptions of Section 2.3), an outcome $a$ is centrality-stable when:

$$k_i a_i = \nu' \left( \sum_{j \in N} a_j \right) \sum_{j \in N} a_j \quad (2)$$

As the right hand side of (2) is the same for all agents, each agent incurs the same total costs (and receives the same total and marginal benefits). This supports our interpretation of (1) as requiring a form of fairness.

Equation (1) demonstrates that at a centrality-stable profile, the vector of contributions is a (right-hand) eigenvector centrality of the gift matrix. While eigenvector centrality conditions have been used in applications such as Google's PageRank measure of websites' importance, and in sociological indices of prestige (Jackson, 2008), to the best of our knowledge we are the first to identify outcomes robust to coalitional deviations in a game using a centrality condition.\(^{10}\) As discussed in the introduction, the other applications of centrality conditions in game theory we are aware of have characterized equilibria in games with only individual deviations and under the restriction that best reply functions are linear. In contrast, we require no parametric restrictions of this type for our result, though we do need some assumptions on the utility functions (recall Section 2.3).

4.3. **Intuition for the Result.** As mentioned in the introduction, we prove Theorem 2 by mapping our repeated game problem into a general equilibrium problem. More specifically, we construct an artificial economy and argue that finding a competitive equilibrium of this economy yields a sustainable action profile. To do this, we first show that outcomes in the core of the stage game are sustainable, under a notion of the core in which any coalitional deviation is automatically and instantaneously punished by non-coalition members reverting to the zero action level. This step builds on a proposition of Aumann (1959). We then recast the static game as a general equilibrium problem with positive externalities. Utilizing ideas developed in Shapley and Shubik (1969) we map

\(^{10}\)Other economic applications in which spectral conditions are found include segregation indices (Echenique and Fryer, 2007) and models of arriving at a consensus through the repeated exchange of information (DeGroot, 1974; DeMarzo, Vayanos, and Zwiebel, 2003).
our problem into a general equilibrium problem without externalities. In this artificial
economy, each agent pays each other agent a price per unit of effort, and agents supply
and demand effort in a competitive market. This market has a competitive equilibrium
and competitive equilibrium outcomes are in the core of the economy.

While the proofs build on standard arguments, throughout we need to do substantial new
work (e.g., use a stronger notion of the core than Aumann used and show it is nonempty)
to construct an equilibrium of our repeated game.

To summarize, the first main insight of this paper is that a coalitionally stable outcome of
the repeated game can be found through conditions that also characterize a competitive
equilibrium of an artificial economy.

The second main insight of the present paper is that, in the environment we study,
a single, simple fixed-point condition characterizes the relevant competitive equilibrium.
In general, an outcome of a competitive economy is specified in terms of prices and
allocations; equilibrium requires, among other things, that each agent be optimizing and
that budgets be balanced. In our setting, the budget-balance conditions alone ensure
that prices can be found that clear a competitive market. The key step in this argument
comes from a novel application (in Section A.3.2 of the appendix) of the Perron-Frobenius
Theorem. This reduction from a complicated condition to a simple one allows both a
conceptual advance – the connection with centrality – and potentially a computational
one, discussed in Section 6.3.

4.4. Scaling-Indifference. Some additional interpretation can be given to Theorem 2
by considering profiles that satisfy scaling-indifference.

Definition. An action profile \( a \in \mathbb{R}^n_{\geq 0} \) satisfies scaling-indifference (or is scaling-indifferent)
if \( a \neq 0 \) and \( J(a)a = 0 \).

By Lemma 1, proved in the appendix, \( a \) is centrality-stable if and only if it satisfies
scaling-indifference. Therefore, an immediate corollary of Theorem 2 is that scaling-
indifferent profiles exist and any such profile is sustainable.

The scaling-indifference condition can be thought of as requiring all agents to be first-
order indifferent to actions being scaled up by a small amount at the margin, which
explains the name. The vector \( J(a)v \) gives the marginal changes in utilities when actions
are changed from \( a \) to \( a + \epsilon v \) for some vector \( v \in \mathbb{R}^n \) and some small real number \( \epsilon \). That
is, to a first-order approximation, \( u(a + \epsilon v) \approx u(a) + \epsilon J(a)v \). Suppose now that actions
are scaled by \( 1 + \epsilon \), for some small real number \( \epsilon \) (this corresponds to setting \( v = a \)).
If \( J(a)a = 0 \), then all agents are indifferent, at the margin, to this small proportional
perturbation in everyone’s actions.\(^{11}\)

Remark 2. Note that the condition \( J(a)a = 0 \) does not change if we form new utility
functions according to the transformation discussed in Remark 1. The Jacobian of the
“new” utilities \( \hat{u} \) is, by the chain rule, simply \( F(u(a))J(a) \), where \( F(u) \) is a diagonal
matrix with \( F_i(u_i) = f_i'(u_i) \). Since \( F(u(a)) \) is nonsingular for the transformation being
considered, it follows that \( J(a)a = 0 \) if and only if \( F(u(a))J(a)a = 0 \). Section 6.2
below discusses reparameterizations of actions rather than utilities. In that case, the
new Jacobian looks like \( J(a) \) multiplied on the right by another matrix, which does
change the set of centrality-stable points.

\(^{11}\)In Section 6.5 it is shown that this can be used to find a gradual path to a sustainable outcome.
5. Private Information

So far we have worked in a setting of complete information. In this section we relax this assumption by considering a specific private information structure we call *hidden benefits*. We first discuss this intuitively and then introduce corresponding formal notions. The setting is characterized by two features:

1. It is common knowledge that each agent’s utility function satisfies Assumptions 1-6 from Section 2.3.
2. If actions are implemented, then the action vector and the Jacobian become common knowledge.

This amounts to assuming that agents can find out marginal costs and benefits of any outcome that actually obtains, but may remain ignorant about other features of the utility functions, especially other agents’ inframarginal costs and benefits.

For an action vector to be sustainable, all agents must be willing to participate. However, under hidden benefits, agents may falsely claim – in order to have a more favorable action vector implemented – that they have a profitable deviation. We suppose that if an action vector is being played, agents can make such false claims as long as the claims are consistent with the information other agents can deduce given (1) and (2) above. Outcomes are quite robust if it is common knowledge that they are sustainable when they are being played, so that such misreporting can never be convincing. On the other hand, if agents can plausibly claim that they can profitably deviate, then an outcome may be upset by such a threat before it is implemented.

We now introduce formal notions to model outcomes robust to misreporting. Given a concave, non-decreasing utility function and an action vector , let be the set of concave utility functions satisfying Assumptions 1, 2, 4, 5, and 6 in Section 2.3 such that for all , we have .

Fix the true utility functions .

**Definition.** An action profile is **non-manipulable** if and only if for all and any , it holds that .

This requires that for any utility function any agent might have given the information publicly available under hidden benefits, she receives a positive payoff. This ensures that no agent can credibly claim to have a profitable individual deviation of playing the zero action. The result of this section is:

**Theorem 3.** Under the assumptions in Section 2.3, an action vector is non-manipulable and Pareto-efficient if and only if it is centrality-stable.

An immediate implication of Theorem 3 is that an action vector is non-manipulable and sustainable if and only if it is centrality-stable. Recall that Theorem 2 guarantees that a centrality-stable action exists and that a action vector is sustainable if it is centrality-stable. Knowing and suffices to verify that a profile is centrality-stable. Thus, under hidden benefits, if a centrality-stable is being played, it is common knowledge that it is sustainable – that no agent or coalition has a profitable deviation. Theorem 3 shows that the centrality-stable are the only action vectors which, when played, are

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12 We are maintaining all the assumptions of Section 2.3, simply replacing the strict concavity assumption with standard concavity, which is all we need for this section.
commonly known to be sustainable under hidden benefits. This is because, according to Theorem 3, any action profile that is not centrality-stable is manipulable. In other words, some agent can claim – consistently with everyone’s knowledge – that his participation constraint would be violated at the proposed action.

6. Discussion

We study repeated games in which agents can take costly actions to create benefits for other agents, and we assume there are positive externalities. In order to do so we introduce two important concepts. The gift matrix describes the marginal benefits each agent can provide to each other agent per unit of marginal costs she incurs. An agent’s contribution is her marginal cost from increasing her action multiplied by her action level. We characterize the Pareto frontier using the spectral radius of the gift matrix: the actions chosen by the players must induce a largest eigenvalue of exactly one in the gift matrix. When an associated right eigenvector of the gift matrix is also equal to agents’ contributions, we call the corresponding action vector centrality-stable. At centrality-stable action profiles, there are no profitable coalitional deviations and these actions are sustainable. Furthermore, centrality-stable action profiles always exist. To the best of our knowledge no other paper has used centrality conditions to describe equilibria robust to coalitional deviations. We are able to do so without making parametric restrictions. Taking a similar approach to analyzing matrices derived from other first order conditions in other settings may prove fruitful.

It seems feasible and worthwhile to extend the analysis to multiple dimensions of effort; this constitutes work in progress. To conclude, we mention some questions for further exploration.

6.1. Stronger Notions of Perfection. As discussed in Section 2.2, the notion of sustainable actions requires rather little from strategies off the equilibrium path – merely that they be part of a subgame-perfect Nash equilibrium, immune to individual deviations. If we envision agents also being able to undertake coalitional deviations after one deviation has occurred, then we should consider different equilibrium notions such as strong perfect equilibrium (Rubinstein, 1980), which is a strategy profile that is a strong Nash equilibrium of every subgame. Unfortunately, as pointed out by Hornaˇ cek (1996, p. 87), in a repeated game with discounting this notion is hopelessly strong: for strictly concave utility functions, it would require that every action profile ever played result in the same payoffs\textsuperscript{13}, which makes punishment impossible.

A way around this problem is to consider a game $\Gamma^*$ defined by a different specification of preferences in the repeated game, namely the Limit of Means Evaluation Relation (LMER): if $(u_{t,i})_{t=0}^\infty$ and $(u'_{t,i})_{t=0}^\infty$ are two (possibly random) streams of payoffs, then we say

$$(u_{t,i})_{t=0}^\infty \succ (u'_{t,i})_{t=0}^\infty \quad \text{iff} \quad \liminf_{T} \sum_{t=0}^{T} \mathbb{E}[u_{t,i} - u'_{t,i}] > 0.$$\textsuperscript{13}

This defines a transitive, though not complete, preference over repeated game payoff streams, and thus over strategy profiles. Given this specification of preferences, we can define LMER-sustainable action profiles as those that are played in some strong perfect

\textsuperscript{13}The key idea is that if, at any history, there are two different stage game payoff vectors that are supposed to be played in the future, the grand coalition would strictly prefer to play a convex combination of those two action vectors.
Nash equilibrium of \( \Gamma^* \). The argument of Rubinstein (1980) shows that these action profiles are actually the same as those that are played in some strong Nash equilibrium of \( \Gamma^* \) – that is, the requirement of strong perfection does not reduce the set of action profiles that can be sustained. The proof of Aumann (1959) shows that these are equivalent to the \( \beta \)-core\(^{14} \) of the stage game; the argument of Section A.3.2 then shows that any centrality-stable vector is in the \( \beta \)-core, and thus is LMER-sustainable. To summarize, under a different utility specification which pays no attention to short-run payoff fluctuations (essentially, no discounting), centrality-stable action vectors are actually LMER-sustainable ones – i.e., they can be sustained in an equilibrium robust to coalitional deviations on and off the equilibrium path.

Returning to the setting with discounting, an alternative way to respond to the potential nonexistence of strong perfect Nash equilibria is to be more demanding of the deviations. Recall that a strategy profile is not a strong perfect equilibrium as long as there is a deviation by some coalition that yields a higher payoff; no strategic coherence of the proposed deviations is needed. As noted earlier, requiring deviations to be robust to further deviations (and thus ruling out some implausible deviations) strictly increases the set of possible equilibria. Following Berheim, Peleg, and Whinston (1987) it is then possible that there exists such an equilibrium (a perfect coalition proof Nash equilibrium) in our environment. Alternatively, one could also define other requirements on deviations, seeking to capture notions such as renegotiation-proofness. We believe it is worth exploring whether this approach can allow the enforcement of certain outcomes with strategies that have a substantial amount of coalitional robustness off the equilibrium path, without the desiderata being so strict that they interfere with equilibrium existence.

Finally, Horniaček (1996) discusses conditions under which equilibria that approximate strong perfect ones exist under discounting; this type of approach may also be fruitfully adapted to our problem.

6.2. **Other Sustainable Outcomes.** Generically, the set of centrality-stable points has measure zero. Thus, Theorem 2 directly identifies a measure zero set of the actions that are sustainable. However, every centrality-stable point \( \mathbf{a} \) is in the relative interior of the set of sustainable outcomes. To see this, consider reparameterizing the action space of each agent, constructing new actions \( \hat{a}_i = g_i(a_i) \) for some smooth, strictly increasing functions \( g_i \). New utility functions are defined by \( \hat{u}_i(\hat{a}_i) = u_i(g_i^{-1}(\hat{a}_i)) \). The main thing we must be careful of is to ensure that this transformation preserves the validity of Assumption 3 (as well as all the other assumptions, but they are preserved automatically).

As the initial utility functions \( u_i \) satisfy our Assumption 3 (strict* concavity) at \( \mathbf{a} \), there exist reparameterizations that preserve this strict* concavity and that implement any small movement of the centrality-stable (and scaling-indifferent) point solving \( \mathbf{J}(\mathbf{a})\mathbf{a} = \mathbf{0} \).

Beyond this, one can consider the class \( \mathcal{C}(\mathbf{u}) \) of all concave utility functions \( \hat{\mathbf{u}} \) that are obtained from \( \mathbf{u} \) by some reparameterizations of the kind described above. Is it the case that any sustainable point corresponds to a centrality-stable point under some such parameterization?

More formally, say that \( \hat{\mathbf{u}} : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}^n \) is equivalent to \( \mathbf{u} \) if, for each \( i \in N \) there is some increasing, continuously differentiable function \( g_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) so that \( \hat{u}_i(\hat{a}_i) = u_i(g_i^{-1}(\hat{a}_i)) \).

\(^{14}\)This is defined like the Pareto \( \beta \)-core in Section A.3.1, except that it is required that every member of the coalition \( M \) benefit strictly from deviating.
\[ u_i^{-1}(\hat{a}_i) \] for all \( \hat{a}_i \in \mathbb{R}_{\geq 0} \). Then let \( C(u) \) be the set of all functions \( \hat{u} \) that are equivalent to \( u \). Also, for any utility function \( \hat{u} : \mathbb{R}_{\geq 0}^n \to \mathbb{R}_{\geq 0}^n \) let \( CS(\hat{u}) \) denote the set of corresponding centrality-stable points.

Is it the case that the set \( R = \{CS(\hat{u}) : \hat{u} \in C(u)\} \) is equal to the set \( S \) of points that are sustainable when utility functions \( u \) define the stage game? The results of this paper show that \( R \) is contained in \( S \). The question of whether the two sets are the same, which to our knowledge is open, seems to be worth investigating.

6.3. Computation. A third issue that may be of concern is computation of centrality-stable outcomes. In general, even verifying that an outcome is sustainable by brute force would require showing that none of the \( 2^n - 1 \) potential deviating coalitions have a profitable deviation. There is evidence that similar computations are generally intractable (Deng and Papadimitriou, 1994). However, to verify that an outcome is centrality-stable (and thus sustainable) requires only the trivial verification of the equation \( J(a)a = 0 \). Finding an action profile that satisfies this equation, rather than merely verifying that a proposed one does, is still nontrivial. In Section 6.4, we discuss a special case in which it is very simple. Smale (1976) gives a general (though sometimes complicated) convergent procedure for finding solutions to many general equilibrium type problems. As discussed in Section 4.3, our conditions simplify matters considerably relative to the typical situation of a competitive equilibrium. Thus, the application of Smale’s techniques to the present setting may yield substantially more direct algorithms and stronger results on their behavior; this is an interesting topic for future study.

6.4. Computation by Gradient Descent. Define the function \( Q : \mathbb{R}_{\geq 0} \to \mathbb{R} \) by
\[
Q(a) = a^T J(a)^T J(a) a.
\]
At any scaling-indifferent outcome \( a \), we have that \( Q(a) = 0 \). The function \( Q \) can sometimes be minimized using a gradient descent algorithm. Setting
\[
W_i(a) = \sum_j a_j \cdot \frac{\partial u_i}{\partial a_j}(a),
\]
it follows that:
\[
Q(a) = \sum_i W_i(a)^2.
\]
If each \( W_i^2 \) is a convex function, then \( Q(a) \) will be convex. It follows that there is precisely one global minimum of this function, which guarantees that a single instance of gradient descent will find a scaling-indifferent (and thus centrality-stable) action profile.

6.5. Approaching a Centrality-Stable Outcome. A final issue is how a sustainable outcome might be implemented in practice if agents are wary of large discontinuous increases in their actions and want each gradual step to benefit them. For example, tariff reductions have in practice been implemented only in a piecemeal way through successive rounds of negotiations. Can actions be increased gradually to a sustainable outcome such that each small increase benefits all agents? The answer turns out to be “yes” for centrality-stable action profiles.

One path to a centrality-stable action profile consists of gradually scaling actions up proportionally to reach the final outcome, making all agents better off with each increase.\(^{15}\)

\(^{15}\)If the agents are countries, then it is plausible that executives (either because of political pressures or for other reasons) are more motivated by short-run considerations than long-run ones.
Formally, fix a centrality-stable outcome \( a \) and imagine a process taking place during times \( t \in [0, 1] \); let us decree that actions at time \( t \) should be \( ta \). To see why all agents always prefer increasing actions as prescribed at time \( t \), assuming everyone else does, consider again the scaling indifference condition \( \textbf{J}(a)a = 0 \). Concavity readily implies that for \( t \in [0, 1) \), \( \textbf{J}(ta) > 0 \), since all entries of the Jacobian of a concave function are nonincreasing in \( t \).\(^{16}\)

An interesting direction for further work would be to study a dynamic game of choosing action profiles over time in the presence of adjustment costs. Do the informal observations above allow us to show that the centrality-stable points are the only ones that can be gradually approached under both farsighted and myopic utility functions in the dynamic game?

\(^{16}\)Footnote 9 shows that at a centrality-stable action profile, all action levels are positive.
References


Appendix A. Proofs

A.1. Preliminaries. Throughout this appendix, we will extensively use the Perron-Frobenius theorem, which we quote here for easy reference. The spectral radius of a matrix is the magnitude (in the complex plane) of all its eigenvalues.

Theorem (Perron-Frobenius). Let $M$ be an irreducible square matrix with no negative entries. Let $r$ be the spectral radius of $M$. Then:

(i) The real number $r$ is an eigenvalue of $M$.
(ii) There is a vector $p$ with only positive entries such that $Mp = rp$.
(iii) If $v$ is a nonzero vector with nonnegative entries so that $Mv = qv$ for some $q > 0$, then $v$ is a positive scalar multiple of $p$, and $q = r$.

Note that because eigenvalues are invariant to taking the transpose of a matrix, all the same statements are true, with the same number $r$, when we replace $M$ by its transpose $M^T$.

The following simple lemmas will also be useful.

Lemma 1. For any $a \in \mathbb{R}_\geq^n$:

(i) the condition $G(a)c(a) = c(a)$ is equivalent to $J(a)a = 0$;
(ii) $G(a)c(a) = rc(a)$ for $r > 1$ implies $J(a)a > 0$; and
(iii) $G(a)c(a) = rc(a)$ for $r < 1$ implies $J(a)a < 0$.

Proof. Let $D(a)$ be a diagonal matrix with $D_{ii}(a) = -J_{ii}(a)$. Dropping the argument $a$ on all the matrices, for all $r > 0$:

$$\begin{align*}
Gc &= rc \\
&\iff (G - I)c = (r - 1)c \\
&\iff JD^{-1}Da = (r - 1)c \\
&\iff Ja = (r - 1)c.
\end{align*}$$

Taking $r = 1$ shows (i); taking $r > 1$ shows (ii); and taking $r < 1$ shows (iii).

We conclude with the simple observation that Assumption 3 implies concavity of each utility function.

Lemma 2. If $u_i$ satisfies Assumption 3, then $u_i$ is concave.

Proof. Take any $a \neq a' \in \mathbb{R}_\geq^n$ and $\lambda \in (0, 1)$. Clearly we can find a sequence $(a(k))_{k=1}^\infty$ converging to $a$ and $(a'(k))_{k=1}^\infty$ converging to $a'$ so that for every $i \in N$ and $k \geq 1$ we have $a_i(k) \neq a_i'(k)$. Let $a''(k) = \lambda a(k) + (1 - \lambda)a'(k)$. By the irreducibility assumption (Assumption 6), there is some $j \in N$ so that $\frac{\partial u_i}{\partial a_j}(a''(k)) > 0$, and thus by Assumption 3 we have $u_i(a''(k)) > \lambda u_i(a(k)) + (1 - \lambda)u_i(a'(k))$. Taking the limit as $k \to \infty$ yields $u_i(a''(k)) \geq \lambda u_i(a(k)) + (1 - \lambda)u_i(a'(k))$. □

17Meyer (2000, Section 8.3) has a statement of the theorem and a comprehensive exposition of it and related results.
A.2. Proof of Theorem 1. Consider the Pareto problem. A social planner can implement any set of actions and places nonnegative Pareto weights \( \theta_i \), not all equal to 0, on different agents' payoffs. The Pareto problem is then:

\[
\max_{a \in \mathbb{R}_{\geq 0}^n} \sum_{i \in N} \theta_i u_i(a).
\]

It is well known that concavity of all the \( u_i \) guarantees that the set of Pareto-efficient points coincides with the set of solutions to this problem for different vectors \( \theta \neq 0 \).

Suppose that an interior action vector \( a^* \) solves this problem. Since it is interior, the solution is characterized by the system of first-order conditions \( \theta J(a^*) = 0 \). Using the definition of \( G \), this is equivalent to each of the following two statements:

\[
\theta (G(a^*) - I) = 0
\]
\[
\theta G(a^*) = \theta
\]

where \( I \) is the identity matrix. Thus \( G(a^*) \) has an eigenvalue of 1. Since \( G(a^*) \) is irreducible, has only nonnegative elements and the eigenvector \( \theta \) is nonnegative, the Perron-Frobenius Theorem guarantees that 1 is a largest eigenvalue of \( G(a^*) \).

Conversely, if \( G(a^*) \) has a largest eigenvalue of 1, then the Perron-Frobenius Theorem guarantees the existence of a nonnegative eigenvector \( \theta \) such that \( \theta G(a^*) = \theta \). By the assumption of concave utilities and the calculation above, it follows that \( a^* \) solves the Pareto problem for Pareto weights \( \theta \).

A.3. Proof of Theorem 2. We first prove assertions (i) and (ii) of the theorem. This proof proceeds in three steps.

1. The set of sustainable action profiles is the same as the strict \( \beta \)-core of the game \( \Gamma \).
2. If \( a \neq 0 \) and \( G(a)c(a) = c(a) \), then \( a \) is in the strict \( \beta \)-core of \( \Gamma \). This, along with (1), implies assertion (i) of the theorem.
3. There is an action vector \( a \in \mathbb{R}_{>0}^n \) such that \( G(a)c(a) = c(a) \). This is assertion (ii) of the theorem.

A.3.1. Step 1. Recall the definition of sustainable actions.

Definition. An action vector \( a \) of the stage game \( \Gamma \) is sustainable if there is a \( \delta < 1 \) so that if \( \delta > \hat{\delta} \), there is a strong Nash equilibrium \( \sigma \) of the repeated game \( \Gamma^*(\delta) \) in which the infinite repetition of \( a \) occurs on the path of play.

Next, we define the strict \( \beta \)-core. An action vector will be in the strict \( \beta \)-core when there is no coalitional deviation that does not leave some member of that coalition worse off, assuming everyone not in the deviating coalition plays the zero action.

Definition. An action profile \( a \) is in the strict \( \beta \)-core of the stage game \( \Gamma \) if it is a Pareto-efficient outcome of the stage game and there is no nonempty proper coalition \( M \subseteq N \) and no other action profile \( a' \) of this game so that:

1. \( a'_i = 0 \) for all \( i \notin M \);
2. each \( i \in M \) weakly prefers \( a' \) to \( a \).
More generally, the $\beta$-core concept\textsuperscript{18} is defined by players outside a deviating coalition punishing the deviating coalition by seeking to minimize their payoffs. In our environment players most severely punish a deviating coalition by choosing an action level of 0.\textsuperscript{19}

**Lemma 3.** An action profile $a$ is sustainable if it is in the strict $\beta$-core of $\Gamma$.

This lemma is related to a proposition in Aumann (1959), which shows that, in a different setting, strong Nash equilibrium payoffs correspond to the (standard) $\beta$-core. The key difference is that Aumann (1959) works under a limit of means evaluation relation (as opposed to discounted payoffs); as discussed below, the “infinite patience” of players in Aumann’s framework makes the result and proof substantially different.

The main idea of the argument is simple: in the strict $\beta$-core concept, we implicitly envision society punishing deviations by taking actions 0. An action profile is in the strict $\beta$-core if and only if a potential deviating coalition is deterred by this “static” punishment in the sense that someone is made strictly worse off. In a repeated game, the punishment of playing 0 in all future periods following a deviation can be implemented. The consequences of that punishment (relative to playing the equilibrium), as evaluated by a potential deviating coalition, are equivalent for $\delta$ close to 1. The proof is nontrivial because, given an action vector in the strict $\beta$-core that we would like to sustain, we need to find a single $\delta < 1$ so that for $\delta \geq \delta$, every potential coalitional deviation available in the dynamic game $\Gamma^*(\delta)$ is simultaneously deterred.

**Proof of Lemma 3.** Fix action vector $a$ in the strict $\beta$-core. We construct a strategy profile which, we claim, enforces repeated play of $a$ as a strong Nash equilibrium of the repeated game $\Gamma^*(\delta)$ (when $\delta \geq \delta$, for a particular $\delta < 1$ to be specified later). Define a strategy profile $\sigma$ in the repeated game at a time $t$ and a given history $(a_1, a_2, \ldots, a_t)$ as follows. For each player: if $a_s$ has been different from $a$ at any $s \leq t$, then play action 0; and otherwise play $a_i$.

We first seek to show that it is sufficient to consider only certain types of coalitional deviations to rule out any coalitional deviation from play of $\sigma$. In particular, we will show that it is without loss of generality to consider coalitional deviations by a nonempty proper subset of agents in which every member of the deviating coalition plays the same action forever after the initial deviation.

Consider now a coalition $M$ that deviates in a (possibly random) period $t$ with a coalitional deviation $\sigma'$. Let the vector $a'_s$ indicate the (possibly random) actions played following the coalitional deviation in period $s > t$ (such that $a_{s,j} = 0$ for all $j \notin M$ as

\textsuperscript{18}The standard $\beta$-core concept is defined similarly to our strict $\beta$-core, except that in the standard $\beta$-core concept (ii) is replaced by the requirement that every member of $M$ strictly prefers the deviation. Our notion is stricter in the sense that the strict $\beta$-core is a subset of the standard $\beta$-core. Note that we also define the strict $\beta$-core in action space rather than, as is usual, in utility space.

\textsuperscript{19}In our environment the $\beta$-core coincides with the $\alpha$-core. The difference between the two is whether the punishment is implemented by non-deviating players in advance of the deviators’ choice or after observing it. To define the $\alpha$-core, we imagine the deviating coalition first choosing their actions to maximize their payoffs and then the other players choosing actions to minimize the payoffs of the deviating coalition given what has happened; to define the $\beta$-core, we imagine the non-deviating players first choosing their actions to minimize the deviating coalition’s payoffs, and then the deviators choosing actions to maximize their payoffs given that. As actions levels of zero always minimize the payoffs of all members of a deviating coalition in $\Gamma$, the order of the moves does not matter.
prescribed by our equilibrium strategies). If $M$ is a proper subset of $N$, define the action vector $\bar{a}$ as follows:

$$\bar{a} \equiv \sum_{s=t+1}^{\infty} \delta^{s-t} E[a'_s].$$

The concavity of the utility functions (see Lemma 2) ensures that the deviation $\sigma''$ which is identical to $\sigma'$ up to and including $t$ and in which agents in $M$ play the constant action $\bar{a}$ afterward is at least as profitable to them as $\sigma'$: For all $i \in M$

$$\sum_{s=t+1}^{\infty} \delta^{s-t} E[u_i(a'_s)] \leq \frac{\delta}{1 - \delta} u_i(\bar{a}).$$

If $M = N$, then define

$$\bar{a} \equiv \sum_{s=t}^{\infty} \delta^{s-t} E[a'_s],$$

noting the change on the indices of summation. Then the same concavity argument shows that if $\sigma'$ is a coalitional deviation for $N$ (i.e. makes every member weakly better off and some strictly better off) then $\bar{a}$ Pareto-dominates $a$ in the stage game $\Gamma$, in contradiction to the definition of the strict $\beta$-core. Thus, from now on we may assume the coalition deviating is a proper subset of agents.

Given these reductions, to show the claim it suffices to show there cannot be a profitable coalitional deviation $\sigma'$ for some proper subset of agents $M$ in which, after the first period in which the action vector does not match $a$, the constant action vector $a'$ is played (with $a'_j = 0$ for $j \notin M$). We will work with such a deviation from now on.

We seek to show that there is at least one member of $M$ (a “scapegoat”) whose payoff in such a deviation is less than her equilibrium payoff. To do this the punishment imposed in the periods following a coalitional deviation must be greater than the one-period gains from deviation (which are bounded, by the assumption that the utility functions $u_i$ are bounded from above). By definition of the strict $\beta$-core, for any $a'$ with $a'_j = 0$ for $j \notin M$, there is some $i \in M$ for whom $u_i(a') < u_i(a)$. However, a concern is that the magnitude of the difference $u_i(a') - u_i(a)$ for this scapegoat $i$ might not be uniformly bounded away from 0 as we range over the set of possible deviation actions $a'$. If this were the case we would not be able to find a fixed $\delta < 1$ (which has to be independent of $a'$, of course) that would make the discounted pain of punishment sufficiently severe no matter what the coalition might do after the deviation. We show now why this problem does not arise.

Let $C$ be the set of nonempty proper subsets of $N$. For $M \in C$, let

$$A_M = \{a' \in \mathbb{R}^n_{\geq 0} : a_i = 0 \text{ for all } i \notin M\}.$$

Recall that we are assuming that after the first period of deviation, a constant action from $A_M$ is played. Define

$$u = \max_{M \in C} \sup_{a' \in A_M} \min_{i \in M} [u_i(a') - u_i(a)].$$

By definition of the strict $\beta$-core, $\min_{i \in N} [u_i(a') - u_i(a)] < 0$ for every $a' \in A_M$. But to avoid the problem we outlined above, we need to ensure that $\min_{i \in M} [u_i(a') - u_i(a)]$ will achieve a maximum over $A_M$, and thus will be uniformly bounded away from 0. Lemma 6 in Step 3 below establishes this through two applications of the fact that a continuous function defined over a compact set achieves a maximum. Thus, every post-deviation
behavior of the coalition results in an amount of per-stage pain strictly bounded away from zero (so the problem we worried about above does not occur).

Define
\[ \bar{p} = \max_{a \in \mathbb{R}^n} \max_{i \in N} \max_{a' \in \mathbb{R}^n} [u_i(a') - u_i(a)]. \]

As all the \( u_i \) are bounded by assumption \( \bar{p} \) is well defined and finite. We can then set\(^{20}\) \( \delta = \bar{p}/(\bar{p} - u) < 1 \) if \( \bar{p} > 0 \), and \( \delta = 0 \) otherwise, and this completes the proof of Lemma 3. \( \square \)

A.3.2. Step 2. Fix \( a^* \in \mathbb{R}^n_{\geq 0} \) such that \( a^* \neq 0 \) and \( G(a^*)c(a^*) = c(a^*) \). We will show that \( a^* \) is in the strict \( \beta \)-core of \( \Gamma \).

First, we will show that \( a^* \) resembles a “market” outcome in the sense that each agent can be thought of as optimizing consumption with certain prices; this will allow us to conclude that the outcome is in the strict \( \beta \)-core by developing techniques familiar from general equilibrium.

Since \( a^* \) is nonzero and Assumption 1 ensures that \( J_{ii}(a) < 0 \) for each \( i \), we know that \( c(a^*) \) is a right eigenvector of \( G(a^*) \) with eigenvalue 1. By the Perron-Frobenius Theorem, there is then a row vector \( \gamma \in \mathbb{R}^n_{\geq 0} \), i.e. one with only positive entries, so that
\[ \gamma G(a^*) = \gamma. \]

From Lemma 1 it follows that \( \gamma J(a^*) = 0 \). Define the (“price”) matrix \( P \) by \( P_{ij} = \gamma_i J_{ij}(a^*) \) and note that for all \( j \in N \) we have
\[ \sum_{i \in N} P_{ij} = \sum_{i \in N} \gamma_i J_{ij}(a^*) = [\gamma J(a^*)]_i = 0, \]
where \([\gamma J(a^*)]_i\) refers to entry \( i \) of the vector \( \gamma J(a^*) \).

Moreover, by Lemma 1, the condition \( G(a^*)c(a^*) = c(a^*) \) implies \( J(a^*)a^* = 0 \) and each row of \( P \) is just a scaling of the corresponding row of \( J(a^*) \). We therefore have:
\[ Pa^* = 0. \]

Consider the following convex program for each of \( i \in M \), denoted by \( \Pi_i \):
\[ \text{maximize } u_i(a) \text{ s.t. } a \in \mathbb{R}^n_{\geq 0} \text{ and } \sum_{j \in N} P_{ij}a_j \leq 0. \]

Intuitively, for \( i \neq j \), the number \( P_{ij} \) can be interpreted as a price that \( i \) pays for the action of \( j \) and \(-P_{ii}\) can be seen as \( i \)'s wage per unit of her own effort. Note that \( \sum_{i \in N} P_{ij} = 0 \) means that agent \( j \)'s wage is equal to the sum of the prices others pay for her effort. The program \( \Pi_i \) captures that each agent seeks to maximize her utility subject to her budget constraint. The proof now follows the standard argument showing the core property of a market equilibrium.

We claim that, for each \( i \), the vector \( a^* \) solves \( \Pi_i \). This is because the gradient of \( u_i \) at \( a^* \), which is row \( i \) of \( J(a^*) \), is normal to the constraint set by construction of \( P \), and by (4) above, \( a^* \) satisfies the constraint. The claim then follows by the concavity of \( u_i \) and the theory of elementary convex programming.

We now use these properties of \( a^* \) to show that it is in the strict \( \beta \)-core of \( \Gamma \). Towards a contradiction, suppose \( a^* \) is not in the strict \( \beta \)-core, and therefore there exists an \( a' \)

\(^{20}\)Recall that \( u < 0 \), so \( \bar{p} - u > \bar{p} \).
(with $a'_i = 0$ for $i \not\in M$) for which $u_i(a') \geq u_i(a^*)$ for each $i \in M$. Since $a^*$ solves the convex program $\Pi_i$, we must therefore have $\sum_{j \in N} P_{ij} a'_j \geq 0$ for $i \in M$.\footnote{Suppose $\sum_{j \in N} P_{ij} a'_j < 0$ for $i \in M$. It follows that, while satisfying the assumption $\sum_{j \in N} P_{ij} a'_j \leq 0$, every $a_j$ for $j \neq i$ can be increased slightly; by Assumption 6, this makes $i$ better off.}

There are then two cases to consider. Suppose first that there is some $i \in M$ such that $u_i(a') > u_i(a^*)$ and so $\sum_{j \in N} P_{ij} a'_j > 0$. If this is true, then:

$$\sum_{i \in M} \sum_{j \in M} P_{ij} a'_j > 0. \quad (5)$$

On the other hand,

$$\sum_{i \in M} \sum_{j \in M} P_{ij} a'_j = \sum_{j \in M} a'_j \sum_{i \in M} P_{ij} \leq \sum_{i \in N} a'_j \sum_{i \in N} P_{ij} = 0. \quad (6)$$

The first equality follows simply from algebraic manipulation, the second inequality holds because $P_{ij} \geq 0$ for $j \neq i$ and the final equality is due to equation (3), which showed that $\sum_{i \in N} P_{ij} = 0$. Equation (6) contradicts equation (5).

We now consider the second case. Suppose that for all $i \in M$, $u_i(a') = u_i(a^*)$. Instead of (5) it now only follows that:

$$\sum_{i \in M} \sum_{j \in M} P_{ij} a'_j \geq 0. \quad (7)$$

We have already considered the case where this inequality holds strictly so suppose that (7) holds with equality. This means that $a'$ is within the constraint set of the convex program $\Pi_i$. Recall $a' > 0$ (recall footnote 9) while $a'_j = 0$ if $j \not\in M$. Fix any such $j$ and any $\lambda \in (0, 1)$ and let $a'' = \lambda a + (1 - \lambda) a''$. By Assumption 6, there is some $i \in M$ so that $\partial u_i(a'')/\partial a_j > 0$, and clearly $a'_j \neq a''_j$. Thus, by Assumption 3 (strict* concavity) we conclude that $u_i(a'') > \lambda u_i(a^*) + (1 - \lambda) u_i(a') = u_i(a^*)$. Moreover, since (7) holds with equality and $a^*$ also satisfies the constraint of $\Pi_i$, it follows that $a''$ satisfies the constraint of $\Pi_i$ as well. But then $a^*$ was not a solution to $\Pi_i$, since $a''$ yields strictly more utility for $i$ and is feasible. This is a contradiction.

A.3.3. Step 3. Before proceeding to the formal proof, we give an intuitive reason for the existence of a centrality-stable $a$. This reason is that, as seen in Section A.3.2 above, the condition $J(a)a = 0$ corresponds to the first-order conditions describing a market equilibrium, and $J(a)a = 0$ is equivalent to $a \neq 0$ being centrality-stable. So we need only to show that a market equilibrium of the type described there exists. Our setting is not a standard private ownership economy to which a standard equilibrium existence theorem applies. However, using ideas of Shapley-Shubik (1969) we can imagine the following useful artificial economy. Give each agent $j$ a large endowment of labor, called good $j$. Introduce a production process that takes one unit of any good $j$ and simultaneously produces many goods: one unit of good $ij$, for every $i \neq j$. The good $ij$ is “$j$’s labor earmarked for $i$”, and is valued only by $i$. These goods are freely traded. An agent’s utility – a function of the goods she buys from others and the amount of labor she puts in – mimics that of the game. Now the economy is standard and a market equilibrium exists (McKenzie, 1959). Since equilibria are efficient, in any equilibrium, every unit of good $ij$ that is valued by $i$ is delivered to that person; units that are not valued at the margin are priced at zero and can be delivered without changing either utilities nor expenditures. So we may assume that, for a fixed $j$, each agent $i \neq j$ receives the same...
quantity of good \( ij \). Thus, the equilibrium of the artificial economy corresponds to some action vector in the original game.

Thus, the claim that there is a centrality-stable action profile essentially reduces to the existence of a competitive equilibrium in a certain artificial economy. However, it is cumbersome to formalize this mapping to this artificial economy and to verify that an equilibrium yields a centrality-stable action profile even in the corner cases. So instead we will show existence directly via a fixed-point argument.

Recall that for this step we must only find a nonzero \( a \) such that \( J(a)a = 0 \). As long as such a point exists we will have (constructively) proved existence of a sustainable outcome by Steps 1 and 2 above. To prove the existence of such an \( a \) we will have (constructively) proved existence of a sustainable outcome.

For any \( i \in N \) and \( a \in \mathbb{R}_{>0}^n \), define \( d_i(\xi; a) \) as the derivative of the function \( t \mapsto u_i(ta) \) in the variable \( t \) evaluated at \( t = \xi \).

Note that by the chain rule, \( d(\xi; a) = J(\xi a)a \). Finally, define \( z : \mathbb{R}_{>0}^n \to \mathbb{R} \) by \( z(a) = \inf \{ \xi \geq 0 : \min_i d_i(\xi; a) \leq 0 \} \). By Assumption 5, the function \( z \) is well-defined (i.e. finite everywhere). It is continuous by the Theorem of the Maximum, since the \( u_i \) are assumed to be continuously differentiable.

We can now define our first key mapping, from the simplex into \( \mathbb{R}_{>0}^n \). This mapping is just a scaling up of actions \( \hat{a} \in \Delta \) by the scalar \( z \) defined above. Specifically, \( h : \Delta \to \mathbb{R}_{>0}^n \) is defined by

\[
    h(\hat{a}) = z(\hat{a}) \hat{a} = a.
\]

At this vector of action levels \( a \) we then construct the gift matrix \( G(a) \). By the Perron-Frobenius Theorem, there is a unique vector \( w(a) \in \mathbb{R}_{>0}^n \cap \Delta \) (i.e., one with only positive entries that sum up to 1) that satisfies \( G(a)w(a) = r(a)w(a) \), where \( r(a) \) is the spectral radius of \( G(a) \). The function \( w : \mathbb{R}_{>0}^n \to \mathbb{R}_{>0}^n \) is continuous (Wilkinson, 1965, pp. 66–67).

We now define our second key mapping back to the simplex. This function takes actions in \( \mathbb{R}_{>0}^n \) (which we will later set to be the ones output by \( h \)), finds the vector \( w(a) \) and then adjusts these values in a particular way and normalizes them. The function \( x : \mathbb{R}_{>0}^n \to \Delta \) is defined by

\[
    x_i(a) = \frac{w_i(a)/J_{ii}(a)}{\sum_{i=1}^n w_i(a)/J_{ii}(a)},
\]

which is positive as \( w_i(a) > 0 \) and \( J_{ii}(a) < 0 \) (by Assumption 1) for all \( i \in N \). It is also continuous since we assumed the \( u_i \) are continuously differentiable.

Consider now the function \( f : \Delta \to \Delta \) defined by

\[
    f(\hat{a}) = x(h(\hat{a}))
\]

Note that for any \( \hat{a} \in \Delta \), we have \( f(\hat{a}) \in \Delta \) by the definitions of \( h, w \) and \( x \). Further, as \( \Delta \) is compact and convex and \( f \) is a continuous function, there exists a fixed point of \( f \) by the Brouwer Fixed Point Theorem. We now just need to show that if we take a fixed
Lemma 4. For any \( \hat{a} \in \Delta \) such that \( f(\hat{a}) = \hat{a} \), we have \( h(\hat{a}) \neq 0 \).

Proof. Assume, toward a contradiction, that \( a = 0 \). Recall that \( h(\hat{a}^*) = z(\hat{a}^*)\hat{a}^* \). As \( \hat{a} \in \Delta \), it follows that \( \hat{a} \neq 0 \), and so \( a = 0 \) implies \( z(\hat{a}) = 0 \). As \( \hat{a} \) is a fixed point of \( f \), we have \( \hat{a} = x(0) \), and we have that \( z(x(0)) = 0 \). By the definition of \( z \), there is then an agent \( i \) for whom \( d_i(0; x(0)) = 0 \).

On the other hand, by the chain rule, \( d(0; x(0)) = J(0)x(0) \). From the definition of \( x \), we have \( x_i(0) = -\alpha(w_i(0)/J_{ii}(0)) \) for some constant \( \alpha > 0 \) (recall \( J_{ii} < 0 \)). Thus, from the definition of \( G \), we deduce \( J(0)x(0) = \alpha(G(0)-I)w(0) \). Further, by Lemma 7 in Section A.5 below, the spectral radius of \( G(0) \) is greater than 1 and so \( G(0)w(0) > w(0) \). But this implies that \( J(0)x(0) > 0 \) (as \( w(0) > 0 \)) and so for all \( i \), \( d_i(0; x(0)) > 0 \). This is a contradiction. \( \square \)

Now we show that \( J(a^*)a^* = 0 \). Recall that we defined the contribution vector so that \( c_i(a) = -a_i \frac{\partial u_i(a)}{\partial a_i} = -a_iJ_{ii}(a) \).

Lemma 5. If \( f(\hat{a}) = \hat{a} \) and \( a = h(\hat{a}) \), then \( G(a)c(a) = r(a)c(a) \), where \( r(a) \) is the spectral radius of \( G(a) \).

Proof. We claim:

\[
 f(\hat{a}) = \hat{a} \implies w(a) = qc(a) \quad \text{for some} \ q > 0. \tag{8}
\]

If this is shown, then since \( G(a)w(a) = r(a)w(a) \) by definition of \( w \), we can conclude \( G(a)c(a) = r(a)c(a) \).

Let us prove (8). Recall that \( a = h(a) \), so that \( a \) is just a scalar multiple of \( \hat{a} \). By Lemma 4, we have \( a = t\hat{a} \) for some \( t > 0 \). From the fixed point condition \( f(\hat{a}) = \hat{a} \) and the definition of \( f \), it follows that:

\[
 \hat{a}_i = f_i(\hat{a}) = x_i(a) = \frac{w_i(a)/J_{ii}(a)}{\sum_{i=1}^{n} w_i(a)/J_{ii}(a)}.
\]

Thus,

\[
 w_i(a) = t^{-1}a_iJ_{ii}(a)\sum_{i=1}^{n} \left( \frac{w_i(a)}{J_{ii}(a)} \right).
\]

To complete the proof of the Lemma and show that \( w_i(a) = qa_iJ_{ii}(a) = qc_i(a) \) we set

\[
 q = t^{-1}\sum_{i=1}^{n} \left( \frac{w_i(a)}{J_{ii}(a)} \right),
\]

which is clearly positive. This completes the proof. \( \square \)

Lemmas 4 and 5, along with the irreducibility of \( G(a^*) \) (recall Assumption 6) ensure that \( a^* > 0 \) – that is \( a^*_i > 0 \) for each \( i \).

Lemma 5 shows that

\[
 G(a^*)c(a^*) = r(a^*)c(a^*). \tag{9}
\]
To finish Step 3 it suffices to prove \( r({\hat{a}}) = 1 \). To do this, note that Lemma 1 shows that if \( r(a^*) > 1 \), then \( J(a^*)a^* > 0 \) (in all dimensions) and if \( r(a^*) < 1 \), then \( J(a^*)a^* < 0 \).

However, from the definition of the function \( z \), the fact that fixed point \( {\hat{a}} > 0 \), and continuity of the \( d_i \) (recall we assumed continuous differentiability of the \( u_i \)) that:

\[
\min_i d_i(z({\hat{a}}); {\hat{a}}) = 0. \quad (10)
\]

Since \( d(0; a^*) = J(a^*)a^* \) by the chain rule, we must therefore have \( r(a^*) = 1 \) and so \( G(a^*)c(a^*) = c(a^*) \).

The following lemma uses the notation developed above and is used in the argument of Step 1.

**Lemma 6.** If \( M \) is a nonempty proper subset of \( N \), and \( A_M = \{a' \in \mathbb{R}_+^n : a_i = 0 \text{ for all } i \notin M \} \), then \( \max_{a' \in A_M} \min_i u_i(a') \) exists and is finite.

**Proof.** For every \( a \in \Delta \cap A_M \), let \( C(a) = \{\xi a : 0 \leq \xi \leq z(a)\} \). Define \( \varphi : \Delta \cap A_M \rightarrow \mathbb{R} \) by

\[
\varphi(a) = \sup_{a' \in C(a)} \min_i u_i(a').
\]

By concavity of the \( u_i \), the maximum of \( \min_i u_i(a) \) over the ray \( \{\xi a : \xi \in \mathbb{R}_+\} \) is achieved over \( C(a) \). The Theorem of the Maximum guarantees that \( \varphi \) is continuous in its argument, so \( \max_{a' \in A_M} \varphi(a') \) exists and is finite (because it is the maximum of a continuous function over a compact set). Together these facts imply the result. \( \square \)

Theorem 2(iii). If an action vector is not Pareto-efficient, it is not in the strict \( \beta \)-core by definition. Step 2 above shows that a centrality-stable action profile is in the strict \( \beta \)-core and so the proof is complete. \( \square \)

### A.4. Proof of Theorem 3.

We first show that if \( G(a)c(a) = c(a) \), then \( a \) is non-manipulable and Pareto efficient.

By the Perron-Frobenius theorem, at actions satisfying \( G(a)c(a) = c(a) \) the largest eigenvalue of \( G(a) \) is 1 and actions \( a \) are Pareto efficient. Now we show that actions \( a \) are also non-manipulable. For any \( \tilde{u}_i \in \mathcal{U}(u_i, a) \), we have

\[
\tilde{u}_i(a) \geq \sum_{j \in N} \frac{\partial \tilde{u}_i(a)}{\partial a_j} a_j = \sum_{j \in N} \frac{\partial u_i(a)}{\partial a_j} a_j = 0. \quad (11)
\]

The first inequality comes from the concavity of all \( \tilde{u}_i \in \mathcal{U}(u_i, a) \). The next equality comes from the fact that for any \( \tilde{u} \in \mathcal{U}(u_i, a) \) and \( j \in N \), it holds that \( \frac{\partial u_i}{\partial a_j} = \frac{\partial u_i}{\partial a_j} \). The final equality holds since the condition \( G(a)c(a) = c(a) \) is equivalent to the condition \( J(a)a = 0 \) (see Lemma 1).

Equation (11) therefore shows that if actions satisfy \( G(a)c(a) = c(a) \), then they are non-manipulable.

Suppose now \( a \) is non-manipulable and Pareto-efficient. Then \( G(a)c(a) = c(a) \).

For any fixed actions \( a \), define the function \( \tilde{u}_i \) by \( \tilde{u}_i(x) := \sum_j J_{ij}(a)x_j \). Note that, for each \( i \), we have \( \tilde{u}_i \in \mathcal{U}(u_i, a) \). Non-manipulability implies that for all \( i \):

\[
\sum_{j \in N} \frac{\partial u_i(a)}{\partial a_j} a_j \geq 0
\]
In matrix notation, this condition is $J(a)a \geq 0$. We claim that in fact $J(a)a = 0$. Suppose not. Then, by the chain rule, for sufficiently small $\epsilon > 0$ it holds that $u_i((1+\epsilon)a) \geq u_i(a)$, with strict inequality for some $i$. This contradicts Pareto-efficiency. Thus, $J(a)a = 0$, which is equivalent to $a$ being centrality-stable by Lemma 1.

A.5. A Technical Lemma. This section is devoted to a simple technical lemma.

Lemma 7. The spectral radius of $G(0)$ is greater than 1.

Proof. By Assumption 4, there is an $a' \in \mathbb{R}_{\geq 0}$ such that $u_i(a') \geq u_i(0)$ for each $i$, with strict inequality for some $i$. Using Assumption 6, namely the irreducibility of $G(a')$, as well as the continuity of the $u_i$, we can find an $a''$ with all positive entries so that $u_i(a'') > u_i(0)$ for all $i$. Let $d$ be the derivative of $u(\zeta a'')$ in $\zeta$ evaluated at $\zeta = 0$. This derivative is strictly positive in every entry, since (by convexity of the $u_i$) the entry $d_i$ must exceed $[u_i(a'') - u_i(0)]/a_i''$. By the chain rule, $d = J(0)a''$. From the fact that this vector is positive, it is immediate to deduce that there is a positive vector $w$ so that $G(0)w > w$, from which it follows by the Collatz-Wielandt formula (Meyer, 2000, equation 8.3.3) that the spectral radius of $G(0)$ exceeds 1. □

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22Let $H(a)$ be a directed graph on the vertex set $N$ defined by having a directed edge $(i, j)$ if and only if $G_{ij}(a) > 0$. By continuity of $G$ in its argument, $H$ is constant in some neighborhood $B$ around $a'$. Now, start at $a(0) := a'$. Given $a(k)$, let $M(k) = \{i : u_i(a(k)) > u_i(a')\}$ and obtain $a(k+1) \in B$ by slightly increasing the actions of all agents in $M(k)$. Do this so that they all remain strictly better off than at $a'$. Note that for all $j$ such that $(i, j) \in H(a(k)) = H(a')$ for some $j \in M(k)$, we have $u_i(a(k+1)) > u_i(a(k)) \geq u_i(a')$. By irreducibility of $G(a')$, the graph $H(a')$ is strongly connected, and so this process will make everyone better off than at $a'$ in a finite number of steps. By continuity of the $u_i$, we can perturb the final $a(k)$ so that all its entries are positive without changing the fact that all agents are better off than at $a'$. 

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