Long Run Optimal Contracts under Adverse Selection with Limited Commitment

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Abstract: The paper studies long run optimal contracts under adverse selection with limited commitment so that the contracts are open to negotiation in every period. Thus the contracting game is repeated over multiple periods and belief about the type of the agent is updated by the principal. We study both the finite-horizon case as well as the infinite-horizon case and find that the unique perfect Bayesian equilibrium of the finite-horizon repeated contracting game is one in which the second-best optimal contract is offered in period 1, there is full revelation of the type of the agent, and from period 2 onwards the first-best contract is offered by the principal. If the agent is the least efficient type then the agent gets no informational rent but if the type of the agent is among the more efficient types, the agent receives an informational rent that has to be paid in period 1. By contrast the infinite-horizon game has multiple Perfect Bayesian Equilibrium points and the one that is optimal for the principal is the equilibrium in which the principal offers the type-separating second-best optimal contract in period 1, fully updates beliefs about the type of the agent and continues to offer the second-best contract from period 2 onwards.

Keywords: Adverse Selection, Optimal Contracts, Limited Commitment, Pooling Contracts, Separating Contracts, Perfect Bayesian Equilibrium, Relational Contract.

JEL Classification Numbers: Primary D2, D8, L1. Secondary L5

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1 Introduction

We study long run contracts when there is limited commitment in an adverse selection model. While enforceability of the contracts is a fairly reasonable assumption in the short run and especially when the contracts are single-period contracts, it is less likely that individuals may sign contracts that are fully enforceable in the long run, and even if they do guaranteeing the enforceability of such contracts may be problematic. What might typically happen when the interaction between a principal and an agent is repeated over many periods, is that the contracts may be rewritten every period, with the contract being enforceable for that period. This of course does not preclude the possibility that the same contract may be offered over multiple periods, but the fact that the individuals are not committed to a single fixed contract over periods allows possibly a greater degree of flexibility.

In order to study the nature of long run contracts with limited commitment we study a repeated game in which a principal and an agent are able to rewrite the contract every period if they choose to do so and only single-period contracts are enforceable. We thus use the framework of repeated games and study a repeated game in which the principal offers an enforceable contract to the agent each period, but the principal is free to offer a different contract in the following period. We find this approach both interesting and useful because it opens up the possibility of renegotiation, as well as allow the principal to offer contracts that are bases on the updated beliefs of the principal. Such an optimal contract problem can then be analyzed as a repeated game with incomplete information in which the payoffs of the players are private information. In the adverse selection models the cost parameter of the agent is not known to the principal and thus the principal does not know the payoff of the agent. Such repeated games have been studied in the literature, especially in the long run pricing and output strategies of firms in oligopoly markets in which the costs of the firms are not known, as for example in [5], [2], [3] and [4]. We find that many of the observations made in these studies are useful in understanding long run optimal contracts in adverse selection models. More specifically we look for contracts that can be implemented as equilibrium points of the
repeated game with incomplete information and focus on the perfect Bayesian equilibrium points of the resulting repeated game with imperfect information.

We find some interesting results. In a finite-horizon repeated game in which both the principal and the agent knows that the relationship between the principal and the agent will end definitely after a given number of periods, the unique perfect Bayesian equilibrium of the game is one in which the optimal second-best contract of the single-period contract problem is offered in period 1. This fully reveals the type of the agent (inferred by the principal from the output produced by the principal) so that the principal updates beliefs about the agent, and then offers the full-information first-best contract that is consistent with the agent’s type to the agent. This, however, means that the principal has to compensate the agent with an informational rent in period 1, so that the agent is willing to reveal his type and forgo the informational rent in the future.

For the infinite-horizon situation we find that there are multiple perfect Bayesian equilibrium points. There is an entire set of perfect Bayesian equilibria that are pooling equilibrium points in which the principal offers a stationary contract in every period and the belief of the principal about the type of the agent is never updated. We also find that there is a perfect Bayesian equilibrium that is a separating equilibrium in which the second-best optimal contract is offered in period 1, the principal’s beliefs are fully updated and the contracts from period 2 onwards is the second-best contract for the type of the agent revealed in period 1. In this separating contract, if the agent is of the more efficient type then the agent gets paid his informational rent every period and thus has no incentive to hide his true type. This particular separating equilibrium is optimal for the principal as the principal’s expected payoff over the entire horizon is a the maximum over all the possible perfect Bayesian equilibrium points. This may at first seem a little counter-intuitive as this does not involve the full-information, first-best contract in any way even though beliefs are fully updated. However, it is useful to note that an information rent has to be paid to the agent in period 1 if the full-information, first-best contract is to be implemented. This information rent is higher than the informational rent of the agent in the second-best optimal contract.
Repeated interaction between a Principal and an Agent is fairly common and much of the negotiation on payments tend to be implicit and have limited commitment over multiple periods. However, although these agreements may not be fully enforceable by long-run contracts, these agreements need to be self-enforcing in some manner. This has been recognized and has been studied in the literature. For example, [14], [13], citeL among others have studied the nature of such self-enforcing or equilibrium contracts with limited commitment in repeated games in the case of imperfect monitoring where the effort level of the agent is unobserved. In some cases the results also extend to the case of adverse selection where the cost parameter of the agent is private information of the agent as in [12]. Studies on adverse selection in the repeated game framework has also been extensive, [6] and [11] study adverse selection with limited commitment in the case of repeated interaction in which the parties can renegotiate a long-term contract. These study long-term contracts in which future renegotiation can be added as a constraint in the initial contract. The result obtained in the case with renegotiation differs significantly from the full-commitment case. Most notably, the renegotiation-proof contract is suboptimal compared to the full-commitment case.

In our study we do not impose any conditions on renegotiation but allow for the possibility that the principal may want to offer different contracts in the future. We examine instead entire classes of self-enforcing contracts in a repeated game framework and examine the properties of these contracts. As we have already mentioned we find that in the finite-horizon game, in which there is a definite terminal point to the relationship between the principal and the agent, there is only kind of contract that is self-enforcing. The principal in this case chooses to learn the type of the agent quickly by paying all the informational rent in period 1 and then offers only the complete information, first-best contract in the following periods.

In the infinite-horizon case we find that there are plenty of self-enforcing series of contracts that can be implemented as either pooling\footnote{We note here that in [12] the contracts in the stationary equilibria are all pooling equilibrium contracts as the contract offered by the principal does not depend on the type of the agent. This is also true of the contracts offered in our pooling equilibria.} or separating equilibrium. However, the series of contracts that the principal would be most
likely to offer is a separating contract as it maximizes the expected stream of profits of the principal. This is equilibrium in which the single-period second-best optimal contract is offered. It is of interest to note that in the case of the pooling equilibria, the agent gains nothing from using his private information and the principal never updates beliefs about the type of the agent. The principal and the agent negotiate only on the available public information about the output level produced. In a separating equilibrium, the principal offers different options to the different types, learns from the output level produced by the agent, who produces an output based on his privately observed cost parameter, and then uses this information to make subsequent offers.

The paper is organized as follows. In section 2 we describe the details of the model. In section 3 we describe the infinite-horizon repeated game with incomplete information. In section 5 we provide a folk theorem for the pooling equilibrium points. In section 6 we discuss the separating equilibrium for both the finite-horizon and the infinite-horizon game. In section 8 we conclude.

## 2 The Adverse Selection Model

A principal needs to contract work out to an agent in which the work needs to be done over many periods. The principal needs to write a contract with the agent in each period although the relationship with the agent can last for many periods. This is typically the situation in many cases where a workers wage or bonuses are determined in each period during which the worker works for the principal. The total revenue of the principal from the output produced by the agent is \( S(q) \) where \( S(\cdot) \) is an increasing and strictly concave function of the output \( q \) produced by the agent.

The agent can produce the output \( q \) at cost \( \theta q \). The value of \( \theta \) is private information to the agent and the principal only knows that \( \theta \) can take finitely many values \( \theta_1, \ldots, \theta_L \) with probabilities \( \nu_1, \ldots, \nu_L \) with \( \theta_1 < \theta_2 < \cdots < \theta_L \). We will denote by \( \Theta = \{\theta_1, \ldots, \theta_L\} \) the set of possible values of \( \theta \) and sometimes refer to \( \Theta \) as the type set of the agent, and the probability distribution giving

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\(^\dagger\)These equilibria are thus similar to the public perfect equilibria, as in [12] in which the strategies of the agent and the principal need only be a function of only the publicly observed output levels.
the belief of the type of the agent we will denote by $\nu$. The principal’s payoff is given by

$$U_P(q, T) = S(q) - T$$

where $T$ is the amount paid by the principal to the agent in return for output $q$. The payoff of the agent which depends on the agent’s type $\theta_\ell$ is given by

$$U_\ell(q, T) = T - \theta_\ell q$$

where $\theta_\ell$ is the true marginal cost of the agent. As the principal does not know the value of $\theta$, the actual payoff of the agent is not known to the principal.

This then gives us the single-period contracting game in which the principal makes an offer $T$ for an output $q$ and the agent then either accepts the offer or rejects it.

3 The infinite horizon game

The infinite horizon repeated adverse selection problem is one that is generated by allowing for recontracting every period over an infinite horizon. The strategy of the principal in this sequential game is a sequence $\{\sigma_t^P\}_{t=1}^{\infty}$ such that

$$\sigma_t^P : H_{t-1} \to \mathbb{R}_+^L \times \mathbb{R}_+^L$$

where $H_{t-1}$ is the set of histories of the game until period $t - 1$ and an $h_{t-1} \in H_{t-1}$ is given by $h_{t-1} = \{(T_1, q_1), (T_2, q_2), \cdots, (T_{t-1}, q_{t-1})\}$, where $q_t$ is the output in time period $t$ and $T_t$ is the payment made in period $t$. That is, $h_{t-1}$ is a history that consists of a sequence of past payments and output levels until period $t - 1$. The menu of choices offered by the principal in period $t$ thus depends on the past history of payments and output levels so that given a history $h_{t-1}$ up to time period $t$, the principal chooses a menu $\sigma_t^P(h_{t-1}) = \{(T_t(\theta_\ell), q_t(\theta_\ell))\}_{\ell=1}^L$ if the principal’s strategy in period $t$ is $\sigma_t^P$. A strategy of the principal will be denoted by $\sigma^P = \{\sigma_t^P\}_{t=1}^{\infty}$. The agent’s strategy in any period $t$ also depends on the past history but also on the type of the agent given by the value of $\theta_\ell$ or the unit cost of production of the agent. Therefore, the strategy of the agent is a sequence $\{\sigma_t^A\}_{t=1}^{\infty}$ such that

$$\sigma_t^A : H_{t-1} \times \Theta \to \mathbb{R}_+.$$
The expected payoff of the principal in period $t$ is

$$\sum_{\ell=1}^{L} \nu_{\ell}(S(q_{\ell}(\theta_{\ell}) - T_{t}(\theta_{\ell}))$$

as the actual payoff of the principal depends on the option in the contract chosen by the agent from the menu offered by the principal. The payoff of the agent in any period $t$ is given by

$$U_{\ell}(q_{t}, T_{t}) = \theta_{t}q_{t} - T_{t}.$$ 

The expected payoff of the principal over the entire infinite horizon is the expected discounted sum of the single-period payoffs from the sequence of offers of the principal and the offers chosen by the agent and is given by

$$\sum_{\ell=1}^{\infty} \nu_{\ell}\left[\sum_{t=1}^{\infty} \delta_{P}^{t-1}(S(q_{\ell}(\theta_{\ell})) - T_{t}(\theta_{\ell}))\right],$$

where $\delta_{P}$ is the discount rate of the principal. Therefore, the expected payoff of the principal when the principal’s strategy is $\sigma^{P}$ and the agent’s strategy is $\sigma^{A\theta}$ is

$$U_{P}^{\infty}(\sigma^{P}, \sigma^{A\theta}) = \sum_{\ell=1}^{\infty} \nu_{\ell}\left[\sum_{t=1}^{\infty} \delta_{P}^{t-1} U_{P}(\sigma_{t}^{P}(h_{t-1}), \sigma_{t}^{A\theta}(h_{t-1}))\right].$$

Similarly, the payoff of the agent over the entire infinite horizon is the discounted sum of the single-period payoffs. These single-period payoffs depend on the offers made each period by the principal and the type of the agent. Thus, the payoff of agent of type $\theta_{\ell}$ over the entire infinite horizon is

$$\sum_{t=1}^{\infty} \delta_{A}^{t-1}(\theta_{t} - T_{t})$$

so that the payoff of the agent over the infinite horizon, when the strategy of the principal and the agent is $(\sigma^{P}, \sigma^{A})$, is given by

$$U_{A}^{\infty}(\sigma^{P}, \sigma^{A}) = \sum_{t=1}^{\infty} \delta_{A}^{t-1} U_{A}(\sigma_{t}^{P}(h_{t-1}), \sigma_{t}^{A}(h_{t-1}))$$

where $\delta_{A}$ is the discount rate of the agent.
4 Long Run Optimal Contracts

In looking for optimal contracts that can be implemented in the long run, that is over the infinite horizon, we look for an optimal contract among the set of equilibrium contracts. While optimal contracts are usually derived by finding the contract that maximizes the principal’s payoff subject to the participation and incentive constraints, in the case of long run contracts with limited commitment where single-period contracts are offered in each period, any long run contract should typically be an equilibrium contract in the sense that the neither the principal nor the agent has any incentive to take an action or make an offer that is different from what is proposed.

In the infinite horizon game with incomplete information the equilibrium concept that we use here is that of a Perfect Bayesian equilibrium. A Perfect Bayesian equilibrium is a strategy combination that continues to be an optimal strategy for every player given any history and the updated beliefs of the players given that history, when the beliefs are updated using Bayes’ rule.

We note that any strategy combination \((σ^P, σ^A)\) generates histories \(h_t\) and thus generates a probability over the set of possible histories \(H_t\) up to time period \(t\). Thus given a strategy combination, observing a history \(h_t\) the principal is able to update beliefs about the type of the agent using Bayes rule, the updated beliefs about the type then is given by the conditional probability distribution over the set \(Θ\) which we will denote as \(ν|(σ, h_t)\).

**Definition 1** Given the strategy combination \(σ^* = (σ^P, σ^A)\), the assessment \((σ^*, ν^*)\) is a Perfect Bayesian equilibrium of the infinite horizon game if

(i) \(ν^*(.)\) is a system of beliefs that is determined by \(σ^*\) according to the Bayesian updating rule, and

(ii) for every time period \(t\) and for every history \(h_t\) up to time period \(t\), the expected payoff of the principal satisfies

\[
\sum_{ℓ=1}^{∞} (ν_ℓ|h_t, σ^*) U^P_∞(σ^P^*, σ^A^*) | h_t) \geq \sum_{ℓ=1}^{∞} (ν_ℓ|h_t, σ^P, σ^A^*) U^P_∞(σ^P, σ^A^* | h_t)
\]

\(‡\)Note that as this is a game with incomplete information it can also be viewed as a game with imperfect information, in which a chance move at the beginning of the game cannot be perfectly observed by all the players.
for every $\sigma^P|_{h_t}$ and the expected payoff of the agent of each type $\ell$ satisfies
\[
\sum_{t=1}^{\infty} \delta_t U^\infty_\ell((\sigma^P, \sigma^{A\ell})|_{h_t}) \geq \sum_{t=1}^{\infty} \delta_t U^\infty_\ell((\sigma^P, \sigma^{A\ell})|_{h_t})
\]
for every $\sigma^{A\ell}|_{h_t}$.

Note that the strategies are conditioned on the private information of the agent as well as the history of outputs and payments. For a detailed discussion of Perfect Bayesian equilibrium and Sequential Equilibrium one may refer to [8] and for a discussion of Sequential equilibrium see [10]. A Perfect Bayesian equilibrium will be called a pooling equilibrium if the strategies of the agent of different types are the same. That is, in a pooling equilibrium the agent plays the same strategy irrespective of its type. A Perfect Bayesian equilibrium will be called a separating equilibrium if the strategy of an agent depends on its type. It will be called a strictly separating equilibrium if the equilibrium strategy of the agent varies strictly with its type.

5 Pooling Equilibrium and a Folk Theorem

Here we show that there are perfect Bayesian equilibrium of the repeated game in which the equilibrium contracts are pooling contracts.

**Theorem 1 A Folk Theorem for Pooling Contracts** Let $(\hat{q}, \hat{T})$ be any pooling contract such that $S(\hat{q}) - \hat{T} > 0$ and $\hat{T} - \theta L \hat{q} > 0$. Then there is a perfect Bayesian equilibrium strategy $\hat{\sigma}$ such that $U^\infty_P(\hat{\sigma}) = \sum_{t=1}^{\infty} \delta_t^{-1} U^\infty(\sigma^P, \sigma^{A\ell})|_{h_t}$ for all $\ell = 1, \ldots, L$.

**Proof:** The claim is that the strategy combination $\{(\hat{\sigma}^P, \hat{\sigma}^{A\ell})|_{h_t}\}$ described below is a pooling equilibrium.

(i) $\hat{\sigma}^P_t(h_{t-1}) = (\hat{q}, \hat{T})$ if the past history has been $(\hat{q}, \hat{T})$ in every period up to $t - 1$.

(ii) If the principal offers $(q, T) \neq (\hat{q}, \hat{T})$ in any period $t$ and both the principal and the agent had offered and produced $(\hat{q}, \hat{T})$ in all previous periods, then the
agent produces \( q = 0 \) from time \( t + 1 \) onwards for \( K \) periods. This is a phase I punishment strategy.

(iii) If the agent produces \( q \neq \hat{q} \) in any period \( t \) and both the principal and the agent had offered and produced \( (\hat{q}, \hat{T}) \) in all previous periods, then the principal offers \( T = \theta_L q \) if \( q \) is the output of the agent for \( K \) periods. This is a phase I punishment for the agent. (iv) If there are no deviations during a phase I punishment by either the principal or the agent then after the length of time \( K \) the principal offers \( (\hat{q}, \hat{T} + \epsilon) \), such that \( \hat{T} + \epsilon < S(\hat{q}) \) if the principal had been the deviator, and offers \( (\hat{q}, \hat{T} - \epsilon) \) such that \( (\hat{T} - \epsilon) - \theta_L q > 0 \) if the agent had been the deviator.

(v) If the agent deviates during a phase I punishment for the principal, then the offer switches to \( T = \theta_L q \) if \( q \) is the output for a length of time \( K \). If the principal deviates while punishing the agent during a phase I punishment then the offer switches to \( q = 0 \) for the next \( K \) periods. Such a punishment is a phase II punishment.

(vi) After a phase II punishment for the principal the offer switches to \( (\hat{q}, \hat{T} + \epsilon) \) and after a phase II punishment for the agent the offer becomes \( (\hat{q}, \hat{T} - \epsilon) \).

(vii) If the principal deviates after a phase II punishment, then the phase I punishment for the principal is played after which the offer becomes \( (\hat{q}, \hat{T} + \epsilon) \).

(viii) Finally, if the agent deviates after a phase II punishment, then the phase I punishment for the agent is played after which the offer becomes \( (\hat{q}, \hat{T} - \epsilon) \).

We now proceed to show that the strategy profile \( \sigma^* \) is an equilibrium irrespective of the type of the agent.

Let \( M_A \) be the maximum “gain” the agent can make by deviating in any period irrespective of its type. If the agent deviates in any period then its maximum payoff in the subsequent periods, if it has cost \( \theta_L \), is at most

\[
M_A + \delta^K \sum_{\nu=1}^{\infty} \delta^{\nu-1} [\hat{T} - \epsilon - \theta_L \hat{q}]
\]

as for a length of time \( K \) the agent’s payoff is zero or less every period. If the agent does not deviate, its payoff in the subsequent periods is

\[
\sum_{\nu=1}^{\infty} \delta^{\nu-1} [\hat{T} - \theta_L \hat{q}].
\]
Therefore, from the construction of the strategy profile, the agent does not gain from a deviation if
\[
\sum_{\nu=1}^{\infty} \delta^{\nu-1}[\hat{T} - \theta_\ell \hat{q}] \geq M_A + \delta^K \sum_{\nu=1}^{\infty} \delta^{\nu-1}[\hat{T} - \epsilon - \theta_\ell \hat{q}].
\] (1)
That is,
\[
\frac{1 - \delta^K}{1 - \delta} [\hat{T} - \theta_\ell \hat{q}] \geq M_A - \frac{\delta^K}{1 - \delta} \epsilon.
\] (2)
Now note that the expression \(\frac{1 - \delta^K}{1 - \delta} \rightarrow K\) as \(\delta \rightarrow 1\), therefore, there is a \(\delta_A : 0 < \delta_1 < 1\) and \(K_1\) sufficiently large for which equation (2) is satisfied for all \(\ell = \{1, \ldots, L\}\). Choose \(K_1\) so that
\[
\frac{1 - \delta^K}{1 - \delta} [\hat{T} - \theta_L \hat{q}] \geq M_A - \frac{\delta^K}{1 - \delta} \epsilon.
\] (3)
Thus, phase I punishments can deter the agent from deviating irrespective of its cost for \(\delta \geq \delta_1\) and \(K \geq K_1\). Similarly, the principal does not gain from a deviation if
\[
\sum_{\nu=1}^{\infty} \delta^{\nu-1}[S(\hat{q}) - \hat{T}] \geq M + \delta^K \sum_{\nu=1}^{\infty} \delta^{\nu-1}[S(\hat{q}) - \hat{T} - \epsilon].
\] (4)
That is, if
\[
\frac{1 - \delta^K}{1 - \delta} [S(\hat{q}) - \hat{T}] \geq M - \frac{\delta^K}{1 - \delta} \epsilon.
\] (5)
Thus, if \(\delta\) is chosen to be sufficiently large (say greater than \(\delta_{P1}\)) and for a large enough \(K\), the principal does not gain from a deviation.

We now consider deviations from a phase I punishment. It should be clear from the above analysis an agent cannot gain while the agent is being punished in a phase I punishment. But consider a deviation made by the agent during a phase I punishment when the principal is considered the deviator. Let \(L_A\) be the maximum loss every period that the agent sustains during a phase I punishment. Then the agent’s payoff after deviating when \(K - t\) (\(1 \leq t < K\)) periods of the phase I punishment is left is then less than or equal to
\[
M_A + \delta^{K-t} \sum_{\nu=1}^{\infty} \delta^{\nu-1}[\hat{T} - \epsilon - \theta_\ell \hat{q}],
\]
and if the agent does not deviate, the payoff in the subsequent periods is:

$$
\delta^{K-t} \sum_{\nu=1}^{\infty} \delta^{\nu-1} [\hat{T} + \epsilon - \theta_{\ell}\hat{q}] - \sum_{\nu=1}^{K-t} \delta^{\nu-1} L_A.
$$

Therefore, the agent does not gain by deviating during a phase I punishment when the principal is being punished, if

$$
\delta^{K-t} \sum_{\ell=1}^{\infty} \delta^{\nu-1} [\hat{T} + \epsilon - \theta_{\ell}\hat{q}] - \sum_{\nu=1}^{K-t} \delta^{\nu-1} L_A \geq M_A + \delta^{K-t} \sum_{\ell=1}^{\infty} \delta^{\nu-1} [\hat{T} + \epsilon - \theta_{\ell}\hat{q}], \quad (6)
$$

This reduces to

$$
\frac{\delta^{K-t}}{1 - \delta} \geq M_A + \frac{1 - \delta^{K-t}}{1 - \delta} L_A.
$$

In equation (7) as $\delta \to 1$, the expression

$$
\frac{1 - \delta^{K-t}}{1 - \delta}
$$

goes to $K - t$ and the expression $\frac{\delta^{K-t}}{1 - \delta}$ goes to $\infty$. Hence, there is a $\delta_{A2} : 0 < \delta_{A2} < 1$ such that equation (7) holds for all $\delta > \delta_{A2}$ and for all $\ell = 1, \ldots, L$.

Again choose $K = K_2$ such that equation (7) holds.

Next, suppose the principal deviates while punishing the agent during a phase I punishment. Then the principal’s payoff from deviating, when $K - t$ ($1 \leq t < K$) periods of the phase I punishment is left, is less than or equal to

$$
M + \delta^{K-t} \sum_{\nu=1}^{\infty} \delta^{\nu-1} [S(\hat{q}) - \hat{T} - \epsilon],
$$

and if the principal does not deviate, the payoff in the subsequent periods is:

$$
\delta^{K-t} \sum_{\nu=1}^{\infty} \delta^{\nu-1} [S(\hat{q}) - \hat{T} + \epsilon] - \sum_{\nu=1}^{K-t} L.
$$

Therefore, the principal does not gain from deviating when the agent is being punished during a phase I punishment if

$$
\delta^{K-t} \sum_{\ell=1}^{\infty} \delta^{\nu-1} [S(\hat{q}) - \hat{T} + \epsilon] - \sum_{\nu=1}^{K-t} L \geq M + \delta^{K-t} \sum_{\ell=1}^{\infty} \delta^{\nu-1} [S(\hat{q}) - \hat{T} - \epsilon]. \quad (8)
$$

\[\text{Note that the type of the agent enters this calculation through } M_A \text{ and } L_A \text{ but these are set so that (6) holds for agents of all types so if } K \text{ is sufficiently large (7) will hold for agents of all types.}\]
This reduces to
\[ \frac{\delta^{K-t}}{1 - \delta} 2 \epsilon \geq M + \frac{1 - \delta^{K-t}}{1 - \delta} L. \]  
(9)

As before, as \( \delta \to 1 \) the left hand side of the inequality in (9) goes to \( \infty \) and the right hand side goes to \( M + (K - t)L \). Hence, there is a \( \delta_{P2} : 0 < \delta_{P2} < 1 \) such that for all \( \delta > \delta_{P2} \) the inequality in (9) holds and the principal cannot gain by deviating during a phase I punishment.

We now consider deviations from a phase II punishment. Consider a deviation by the agent from a phase II punishment while punishing the principal. The payoff of the agent, if the agent deviates after \( t \) periods of the phase II punishment, is at most
\[ M_A + \delta^{K-t} \sum_{\nu=1}^{\infty} \delta^{\nu-1} [\hat{T} - \epsilon - \theta_l \hat{q}], \]
and if the agent does not deviate, the payoff in the subsequent periods is
\[ \delta^{K-t} \sum_{\ell=1}^{\infty} \delta^{\nu-1} [\hat{T} + \epsilon - \theta_l \hat{q}] - \sum_{\nu=1}^{K-t} \delta^{\nu-1} L_A. \]
Therefore, the agent does not gain by deviating after \( t \) periods during a phase II punishment when the principal is being punished, if
\[ \delta^{K-t} \sum_{\ell=1}^{\infty} \delta^{\nu-1} [\hat{T} + \epsilon - \theta_l \hat{q}] - \sum_{\nu=1}^{K-t} \delta^{\nu-1} L_A \geq M_A + \delta^{K-t} \sum_{\nu=1}^{\infty} \delta^{\nu-1} [\hat{T} - \epsilon - \theta_l \hat{q}], \]  
(10)

Note that this inequality is the same as the one in (6) and the same analysis that follows shows that for \( \delta \geq \delta_{A2} \) the agent cannot gain by deviating from a phase II punishment.

Similarly, for the principal, a deviation from a phase II punishment while punishing the agent is not profitable if (8) holds and thus is not profitable for \( \delta \geq \delta_{P2} \).

Finally, we consider deviations from the contracts \((\hat{q}, \hat{T} - \epsilon)\) and \((\hat{q}, \hat{T} + \epsilon)\) respectively. For the agent it is enough to show that the agent cannot profitably deviate from \((\hat{q}, \hat{T} - \epsilon)\). If the agent deviates then the subsequent payoff of the agent of any type is at most
\[ M_A + \delta^K \sum_{\nu=1}^{\infty} \delta^{\nu-1} (\hat{T} - \epsilon - \theta_l \hat{q}) \]
and if he does not deviate then the payoff in the subsequent periods is
\[ \sum_{\nu=1}^{\infty} \delta^{\nu-1}(\hat{T} - \epsilon - \theta \ell \hat{q}). \]

Therefore, the agent does not gain from deviating if
\[ \sum_{\nu=1}^{\infty} \delta^{\nu-1}(\hat{T} - \epsilon - \theta \ell \hat{q}) \geq M_A + \delta^K \sum_{\nu=1}^{\infty} \delta^{\nu-1}(\hat{T} - \epsilon - \theta \ell \hat{q}). \tag{11} \]

This reduces to
\[ \frac{1 - \delta^{K+1}}{1 - \delta} (\hat{T} - \epsilon - \theta \ell \hat{q}) \geq M_A. \tag{12} \]

As \( \delta \to 1 \), \( \frac{1 - \delta^{K+1}}{1 - \delta} \to K + 1 \). Hence, for \( K \) such that
\[ (K + 1)(\hat{T} - \epsilon - \theta \ell \hat{q}) > M_A \]
there is a \( \delta_{A3} \) such that for all \( \delta \geq \delta_{A3} \) the inequality in (11) holds and the agent cannot gain by deviating. A similar analysis for the principal shows that the principal cannot gain from deviating from \((\hat{q}, \hat{T} + \epsilon)\) if
\[ \sum_{\nu=1}^{\infty} \delta^{\nu-1}(s(\hat{q}) - \hat{T} + \epsilon) \geq M + \delta^K \sum_{\nu=1}^{\infty} \delta^{\nu-1}(S(\hat{q}) - \hat{T} + \epsilon) \tag{13} \]
that is if
\[ \frac{1 - \delta^{K+1}}{1 - \delta} (S(\hat{q}) - \hat{T} + \epsilon) \geq M. \tag{14} \]

Hence, there is a \( \delta_{P3} \) such that for all \( \delta \geq \delta_{P3} \) the inequality in (14) will hold if \( K \) satisfies
\[ (K + 1)(S(\hat{q}) - \hat{T} + \epsilon) > M. \]

We have therefore shown that for \( \delta > \max\{\delta_{A1}, \delta_{A2}, \delta_{A3}\} \), the agent cannot gain by deviating in any period \( t \), given any history, and for \( \delta > \max\{\delta_{P1}, \delta_{P2}, \delta_{P3}\} \), the principal cannot gain by deviating in any period \( t \), given any history.

We now show that the strategy combination \{\((\hat{\sigma}^P, \hat{\sigma}^{A\ell})\)\}_{\ell=1}^{L} \} is a Perfect Bayesian equilibrium. We first note that since \( (\hat{\sigma}^P|\hat{h}_t, \hat{\sigma}) = (\hat{\sigma}^P|\hat{h}_t, \hat{\sigma}) \) and \( (\hat{\sigma}^{A\ell}|\hat{h}_t, \hat{\sigma}) = (\hat{\sigma}^{A\ell}|\hat{h}_t, \hat{\sigma}) \) for every \( \ell \), therefore we have \( \nu_t|\hat{h}_t, \hat{\sigma} = \nu_t|\hat{h}_{t-1}, \hat{\sigma} \) for all \( t \geq 1 \). Hence, \( \nu_t|\hat{h}_t, \hat{\sigma} = \nu_t \) for all \( \ell = 1, \cdots, L \) so that
\[ \sum_{\ell=1}^{L} (\nu_t|\hat{h}_t, \hat{\sigma}) U^\infty_P(\hat{\sigma}|\hat{h}_t, \ell) = U^\infty_P(\hat{\sigma}|\hat{h}_t, \ell) \sum_{\ell=1}^{L} \nu_t|\hat{h}_t, \ell \]
\[ = U^\infty_P(\hat{\sigma}|\hat{h}_t, \ell). \tag{15} \]
Since we have already shown that for any $h_t$ and $\ell = 1, \ldots, L$ and for all strategy $\sigma^P$ of the principal, $U_P^\infty(\hat{\sigma}|h_t, \ell) \geq U_P^\infty((\sigma^P, \hat{\sigma}^A)|h_t, \ell)$, it now follows from (15) that
\[
\sum_{\ell=1}^{L} (\nu|_{h_t, \hat{\sigma}}) U_P^\infty(\hat{\sigma}|h_t, \ell) = U_P^\infty(\hat{\sigma}|h_t, \ell) \geq \sum_{\ell=1}^{L} (\nu|_{h_t, \hat{\sigma}}) U_P^\infty((\sigma^P, \hat{\sigma}^A)|h_t, \ell). \tag{16}
\]

Similarly, as we have already shown that for any $h_t$ and for all strategy $\sigma^A_\ell$ of the agent with cost $\theta_\ell$, $U_{A_\ell}^\infty(\hat{\sigma}|h_t, \ell) \geq U_{A_\ell}^\infty((\hat{\sigma}^P, \sigma^A_\ell)|h_t, \ell)$, for all $\ell = 1, \ldots, L$ it follows from $\nu|_{h_t, \hat{\sigma}} = \nu_\ell$ for all $\ell = 1, \ldots, L$ that
\[
\sum_{\ell=1}^{L} (\nu|_{h_t, \hat{\sigma}}) U_{A_\ell}^\infty(\hat{\sigma}|h_t, \ell) = U_{A_\ell}^\infty(\hat{\sigma}|h_t, \ell) \geq \sum_{\ell=1}^{L} (\nu|_{h_t, \hat{\sigma}}) U_{A_\ell}^\infty((\hat{\sigma}^P, \sigma^A_\ell)|h_t, \ell). \tag{17}
\]
But (16) and (17) then show that $\hat{\sigma}$ is a Perfect Bayesian equilibrium. It is by construction a pooling equilibrium. This thus concludes the proof.

6 Equilibrium Contracts in the Finite-Horizon

While the result on pooling equilibrium shows that there are plenty of pooling contracts that are perfect Bayesian equilibrium of repeated games between the principal and the agent, we know that the optimal contract for the single-period is a separating contract in the sense that a menu of contracts is offered with each option in the menu meant for an agent of a particular type. Here we show that the only perfect Bayesian equilibrium of the repeated contract game with finite-horizon is the one in which the second-best optimal contract is offered in period 1, and then the first-best complete information contract is offered in the subsequent periods. It is well known that in the single-period, second-best optimal contract, the least efficient type with the highest marginal cost $\theta_L$ is offered $T_L = \theta_L q_L$ for the output $q_L$. That is, the single-period optimal contract is such that
\[
U_L(q_L, T_L) = T_L - \theta_L q_L = 0.
\]
Further, for $\ell \neq L$, the incentive compatible offers all satisfy the condition that

$$U_\ell(q_\ell, T_\ell) = T_\ell - \theta_\ell q_\ell > 0.$$ 

In this standard environment, the first-best outcome, in which the principal maximizes his profit, is characterized by

$$S'(q^*_\ell) = \theta_\ell$$

where the efficient outcome is obtained by equating the principal’s marginal benefit to the agent’s marginal cost. We first examine what happens in a $T$-period contract. For this, we assume the monotone hazard rate property:

$$\sum_{i=1}^{\ell-1} \nu_k < \sum_{k=1}^{\ell} \nu_k < \sum_{k=1}^{\ell+1} \nu_k$$

for all $\ell = 1, \ldots, L - 1$.

From the literature on optimal contracts we know that this assumption implies that the second-best single-period contract will fully separate types in the sense that, in the optimal menu, the output levels assigned to the different types will be distinct; that is there will be no bunching.

Recall that when the principal offers the menu of contracts, it has to satisfy the following constraints:

(i) $T_\ell - \theta_\ell q_\ell \geq T_k - \theta_k q_k$ for all $\ell, k$ and

(ii) $T_\ell - \theta_\ell q_\ell \geq 0$ for all $\ell$.

The first constraints (i) are the incentive compatibility constraints. The second constraints (ii) are participation constraints. This problem can be reduced greatly as follows. First, as in the two-type case, the least efficient agent’s participation constraint is binding because $T_\ell - \theta_\ell q_\ell \geq T_k - \theta_k q_k \geq T_L - \theta_L q_L \geq 0$.

Second, since the agent’s utility function meets the single-crossing conditions,

$$\frac{\partial}{\partial \theta} \left( -\frac{\partial U/\partial q}{\partial U/\partial T} \right) = 1 > 0,$$

we can impose the monotonicity constraints such as $q_\ell \geq q_2 \geq \cdots \geq q_L$. In addition, we can restrict our attention on the local incentive constraints between two adjacent types and these incentive constraints will bind at the optimum.

\footnote{Consider the following the incentive constraints for $\theta_\ell \neq \theta_k$; $T_\ell - \theta_\ell q_\ell \geq T_k - \theta_k q_k$ and $T_k - \theta_k q_k \geq T_\ell - \theta_\ell q_\ell$. Summing the two constraints, we obtain $(\theta_\ell - \theta_k)(q_\ell - q_k) \geq 0$.}

\footnote{Any global incentive constraint is implied by the two local incentive constraints. Consider $\theta_i < \theta_k < \theta_j$. The incentive constraint between $\theta_i$ and $\theta_k$ is $T_i - \theta_i q_i \geq T_k - \theta_k q_k$. Similarly, the incentive constraint between $\theta_k$ and $\theta_j$ is $T_k - \theta_k q_k \geq T_j - \theta_j q_j$. Adding the two constraints, we obtain $T_i - \theta_i q_i \geq T_j - \theta_j q_j + \theta_k(q_k - q_j)$. It is immediate that $T_i - \theta_i q_i \geq T_j - \theta_j q_j$.}
The reduced problem is given by

\[
\text{Maximize}_{\{q_\ell, T_\ell\}} \sum_{l=1}^{L} v_l(S(q_\ell) - T_\ell)
\]
such that

(i) \( T_\ell - \theta_\ell q_\ell = T_{\ell+1} - \theta_{\ell} q_{\ell+1} \) for \( \ell = 1, \ldots, L - 1 \),

(ii) \( T_L - \theta_L q_L = 0 \),

(iii) \( q_1 \geq q_2 \geq \cdots \geq q_L \).

The analysis is a quite straightforward extension of the standard two-type model. There is no distortion on the most efficient type agent’s output. However, for the less efficient types, the production levels are distorted downward.

We turn to the next benchmark in which the principal makes a long term contract with the agent. In a dynamic model, the commitment is an important issue. The contractual outcome would be very different if the principal is not able to commit to renegotiating the initial contract. The reason is well-known in the literature. When the type of the agent is revealed in the first period, the principal can exploit this information fully if renegotiation is feasible. Below, we allow renegotiation in any period and investigate the renegotiation-proof contracts.

Denote the contract offered to type \( \ell \) in period \( t \) by \((T_\ell,t, q_\ell,t)\). Suppose that the principal offers a separating contract for the first time in period \( b_t \). Thus from periods 1 through \( b_t - 1 \), the principal offers a pooling contract. After the separating contract in period \( b_t \), the principal can offer the complete-information first-best contract.

For the pooling contract in period \( t \leq b_t - 1 \), we assume that the principal wants to induce the participation of the least efficient agent. Then, the pooling contract is simply \( q_L^* \) so that \( T_{L,t}^* - \theta_L q_{L,t}^* = 0 \). After separation in period \( b_t \), the first-best contract can be offered because the type of the agent is fully revealed so that \( t_\ell = \theta_\ell q_\ell^* \) for all \( \ell \). This fully separating contract is robust to the possibility of renegotiation because there does not exist a Pareto-improving contract. Thus, we can apply the revelation principal to find the separating contract in period \( t = b_t \).
Note however, that in order to induce the agent to produce the level of output that would reveal his true type, the principal has to compensate the agent for the revelation of his type. Thus, to induce information revelation in period $\hat{t}$, type-$\ell$ agent’s intertemporal incentive constraint has to satisfy

$$T_{\ell,\hat{t}} - \theta_\ell q_{\ell,\hat{t}} \geq T_{\ell+1,\hat{t}} - \theta_\ell q_{\ell+1,\hat{t}} + \sum_{t=\hat{t}+1}^{T} \delta^{t-\hat{t}} \left( T_{\ell+1,t}^* - \theta_\ell q_{\ell+1,t}^* \right).$$

Note that from period $\hat{t}$, the agent’s type is fully revealed and so each type of agent receives zero rent onward until $T$, i.e., $T_{t,\ell}^* = \theta_\ell q_{\ell,t}^*$. In the long term contract with finite periods, a difference is that type $\ell < L$ agent will get an additional rent $\sum_{t=\hat{t}+1}^{T} \delta^{t-\hat{t}} (\theta_{\ell+1} - \theta_\ell) q_{\ell+1,t}^*$. This tightens the IC constraints, but does not affect the other constraints. Thus, in $t = \hat{t}$, the principal’s problem is given by

$$\max_{\{q_{t,\ell}, T_{t,\ell}\}} \sum_{t=\hat{t}}^{T} \delta^{t-\hat{t}} \sum_{\ell=1}^{L} \nu_\ell \left( S(q_{\ell,t}) - T_{t,\ell} \right)$$

such that

(i) $T_{\ell,\hat{t}} - \theta_\ell q_{\ell,\hat{t}} = T_{\ell+1,\hat{t}} - \theta_\ell q_{\ell+1,\hat{t}} + \sum_{t=\hat{t}+1}^{T} \delta^{t-\hat{t}} \left( T_{\ell+1,t}^* - \theta_\ell q_{\ell+1,t}^* \right)$

(ii) $T_{L,\ell} - \theta_L q_{L,\ell} = 0$, and

(iii) $q_{1,t} \geq q_{2,t} \geq \cdots \geq q_{L,t}$.

**Lemma 1** In period $t = \hat{t}$, the optimal separating contract is the second-best contract given by

$$q_{1,\hat{t}} = q_1^* \text{ and } q_{\ell,\hat{t}} = q_{\ell}^{SB} \text{ for } \ell \geq 2.$$

The proof of this lemma is omitted because it is a straightforward extension of the single period case. The principal’s expected intertemporal profit can now be written as

$$\sum_{t=1}^{T} \sum_{\ell=1}^{L} \nu_\ell U_P(\cdot; \hat{t}) = \sum_{t=1}^{\hat{t}-1} \{ S(q_{L,t}^*) - \theta_L q_{L,t}^* \}$$

$$+ \delta^{\hat{t}-1} \left\{ \nu_1 (S(q_1^*) - \theta_1 q_1^*) + \sum_{\ell=2}^{L} \nu_\ell (S(q_{\ell}^{SB}) - \theta_\ell q_{\ell}^{SB}) - \left[ \sum_{\ell=1}^{L} \nu_\ell (\theta_{\ell+1} - \theta_\ell) q_{\ell+1}^{SB} \right] \right\}$$

$$- \delta^{\hat{t}-1} \left[ \sum_{t=\hat{t}+1}^{T} \delta^{t-\hat{t}} \sum_{\ell=1}^{L} \nu_\ell (\theta_{\ell+1} - \theta_\ell) q_{\ell+1}^{SB} \right] + \sum_{t=\hat{t}+1}^{T} \delta^{t-\hat{t}} \sum_{\ell=1}^{L} \nu_\ell \{ S(q_{\ell,t}^*) - \theta_\ell q_{\ell,t}^* \}. $$
The first term in the RHS is the principal’s profit from the pooling contract from \( t = 1 \) to \( t = \hat{t} - 1 \). The term in the second line of the RHS is her profit from the separating contract in period \( t = \hat{t} \). The first term in the third line is the informational rent that the principal has to pay for early revelation. Note that the principal has to pay two types of information rent to induce truth-telling. The first is the typical information rent in a single period. The second is the one for early revelation. The last term on the RHS is her profit from the first-best contract from \( t = \hat{t} + 1 \) to \( t = T \).

What would be the optimal timing of separation? The answer is not \textit{a priori} clear because the principal has to compensate a large amount of information rent for the agent’s early revelation.

**Lemma 2** The optimal timing of separation is \( \hat{t} = 1 \).

**Proof:** The principal’s intertemporal profit can be rewritten as

\[
\begin{align*}
\sum_{t=1}^{T} \sum_{\ell=1}^{L} \nu_{t} U_{P}(\cdot \hat{t}) &= \sum_{t=1}^{\hat{t} - 1} \delta^{t-1} \{ S(q_{L}^{*}) - \theta_{L} q_{L}^{*} \} \\
&+ \delta^{\hat{t} - 1} \left\{ \nu_{1} (S(q_{1}^{*}) - \theta_{1} q_{1}^{*}) + \sum_{\ell=2}^{L} \nu_{\ell} (S(q_{\ell}^{SB}) - \theta_{\ell} q_{\ell}^{SB}) - \sum_{t=1}^{L} \nu_{t} (\theta_{t+1} - \theta_{t}) q_{t+1}^{*} \right\} \\
&+ \delta^{\hat{t} - 1} \sum_{t=\hat{t}+1}^{T} \delta^{t-\hat{t}} \left\{ \sum_{\ell=1}^{L} \nu_{t} \left[ (S(q_{t}^{*}) - \theta_{t} q_{t}^{*}) - (\theta_{t+1} - \theta_{t}) q_{t+1}^{*} \right] \right\}.
\end{align*}
\]

Note that the information rent paid by the principal is included in this payoff. However, if the contract is a pooling one \((q, T)\) then the payoff of the principal is simply \((S(q) - T)\). Thus, the principal’s intertemporal profit can now be simply rewritten as

\[
\begin{align*}
\sum_{t=1}^{T} \sum_{\ell=1}^{L} \nu_{t} U_{P}(\cdot \hat{t}) &= \sum_{t=1}^{\hat{t} - 1} \delta^{t-1} (S(q_{L}^{*}) - \theta_{L} q_{L}^{*}) + \delta^{\hat{t} - 1} \tilde{U}_{P}(q_{1}^{*}, q_{2}^{SB}, \ldots, q_{L}^{SB})
\end{align*}
\]
where up to period $\hat{t} - 1$, the principal offers the optimal pooling contract, and then the second-best optimal contract, and then the complete-information separating contract. We claim that

$$
\sum_{t=1}^{T} \sum_{\ell=1}^{L} \nu_t U_P(\cdot; \hat{t}) - \sum_{t=1}^{T} \sum_{\ell=1}^{L} \nu_t U_P(\cdot; \hat{t} + 1) \geq 0.
$$

To see this, observe that

$$
= \delta^{\hat{t}-1} \left[ \sum_{t=1}^{T} \sum_{\ell=1}^{L} \nu_t U_P(\cdot; t) - \sum_{t=1}^{T} \sum_{\ell=1}^{L} \nu_t U_P(\cdot; t + 1) \right]
\geq 0.
$$

Recall that $\tilde{U}_P(q_1, q_2, \ldots, q_L) = \sum_{t=1}^{T} \nu_t [(S(q_t) - \theta_t q_t) - (\theta_{t+1} - \theta_t) q_{t+1}]$ is the principal’s reduced maximization problem in a single period contract after inserting the binding Incentive compatibility constraints and the least efficient agent’s participation constraint into the objective function. As the optimal solution of principal’s constrained optimization problem is $q_1 = q^*_1$ and $q_t = q^*_L$, for $t > 1$ we must have

$$
\tilde{U}_P(q^*_1, q^*_2, \ldots, q^*_L) \geq \max \left\{ \tilde{U}_P(q^*_1, q^*_2, \ldots, q^*_L), \tilde{U}_P(q^*_1, q^*_L, \ldots, q^*_L) \right\}
$$

We further claim that

$$
\tilde{U}_P(q^*_1, q^*_2, \ldots, q^*_L) \geq \tilde{U}_P(q^*_1, q^*_L, \ldots, q^*_L).
$$

Note that

$$
\tilde{U}_P(q^*_1, q^*_2, \ldots, q^*_L-2, q^*_L-1, q^*_L) - \tilde{U}_P(q^*_1, q^*_2, \ldots, q^*_L-2, q^*_L, q^*_L)
= \nu_{L-1} \left[ \left( S(q^*_L-1) - \theta_{L-1} q^*_L-1 \right) \right] \geq 0.
$$

This is because $q^*_L-1 = \arg\max_{q^*_L-1} [S(q^*_L-1) - \theta_{L-1} q^*_L-1]$. It now follows in a similar way that

$$
\tilde{U}_P(q^*_1, q^*_2, \ldots, q^*_L-2, q^*_L-1, q^*_L) \geq \tilde{U}_P(q^*_1, q^*_2, \ldots, q^*_L-2, q^*_L, q^*_L)
\geq \tilde{U}_P(q^*_1, q^*_2, \ldots, q^*_L-3, q^*_L, q^*_L, q^*_L).
$$
Arguing recursively in this it now follows that
\[ \tilde{U}_P(q_1^*, q_2^*, \ldots, q_L^*) \geq \tilde{U}_P(q_L^*, q_L^*, \ldots, q_L^*). \] (21)

Using (21) in (20) we now have
\[
\left[ \sum_{t=1}^{T} \sum_{\ell=1}^{L} \nu_{t} U_P(\cdot; \hat{t}) - \sum_{t=1}^{T} \sum_{\ell=1}^{L} \nu_{t} U_P(\cdot; \hat{t} + 1) \right] \\
= \delta^{\hat{t}-1} \left[ \tilde{U}_P(q_1^*, q_2^{SB}, \ldots, q_L^{SB}) - \tilde{U}_P(q_L^*, q_L^*, \ldots, q_L^*) \right] \\
+ \delta^{\hat{t}} \left[ \tilde{U}_P(q_1^*, q_2^*, \ldots, q_L^*) - \tilde{U}_P(q_1^*, q_2^{SB}, \ldots, q_L^{SB}) \right] \\
\geq \delta^{\hat{t}-1} \left[ \tilde{U}_P(q_1^*, q_2^{SB}, \ldots, q_L^{SB}) - \tilde{U}_P(q_1^*, q_2^*, \ldots, q_L^*) \right] \\
+ \delta^{\hat{t}} \left[ \tilde{U}_P(q_1^*, q_2^*, \ldots, q_L^*) - \tilde{U}_P(q_1^*, q_2^{SB}, \ldots, q_L^{SB}) \right] \\
\geq \delta^{\hat{t}-1} (1 - \delta) \left[ \tilde{U}_P(q_1^*, q_2^{SB}, \ldots, q_L^{SB}) - \tilde{U}_P(q_1^*, q_2^*, \ldots, q_L^*) \right] \\
\geq 0. \] (22)

This proves the claim in (19). Hence, the intertemporal contract that gives the principal the highest expected profit is the one that offers the second-best contract in period 1 together with the informational rents to the types \( \ell > 1 \) and then set output levels at \( q_1^* \) from period 2 onwards.

This result shows that in the contracting game with limited commitment that is repeated for \( T \) periods the only possible equilibrium in the game is one in which the principal offers the separating contract in period 1 together with the extra informational rent, learns about the type of the agent in period 1, and then sets the output at the first-best, efficient level for the type of the agent inferred from the output of the agent in period 1.

**Theorem 2** The unique Perfect Bayesian equilibrium in the \( T \)-period repeated contract game is one in which the principal offers the contract \( \{ q_\ell, T_\ell \}_{\ell=1}^{L} \) in period 1 together with the informational rent \( \frac{\delta(1-\delta^{T-1})}{1-\delta} \) if the agent produces \( q_\ell \) in period 1, and from period 2 onwards sets \( q_\ell = q_1^* \) and \( T_\ell = \theta_1 q_1^* \).

**Proof:** From lemma 2 it follows that the optimal strategy of the principal in the \( T \)-period game is to offer the optimal separating contract in period 1 and then to offer the first-best, full information contract to the agent that is consistent with his choice of output.
The best response of the agent in period 1 to this offer is to produce the output level consistent with its type. The updated belief of the principal is then that \( \text{prob.}(\hat{\ell}) = 1 \) if \( q_1 = q_{\ell}^{SB} \) and \( \text{prob.}(\ell) = 0 \) otherwise. The principal then offers the contract \( (q_\ell = q_{\ell}^1, T_1 = \theta_L q_\ell) \) for \( t \geq 2 \). The agent’s best response given this is to produce the output level \( q_\ell = q_{\ell}^1 \) in every period.

Thus in the standard adverse selection model with limited commitment, the principal prefers to separate in the first stage and then offer the complete-information, first-best contract after that in the finite-horizon case. A pooling contract is never part of a perfect Bayesian equilibrium in the finite-horizon case. This contrasts sharply with theorem 1.

7 Separating Long Run Contracts

The next result shows that in the infinite-horizon game not only is a pooling contract a perfect Bayesian equilibrium but so is the continuation of the second-best contract even after the type of the agent’s type has been fully revealed.

**Theorem 3** There is a perfect Bayesian equilibrium in which the optimal single-period contract is offered in period 1, and from period 2 onwards, the only contract offered in equilibrium is \( (T_\ell, q_\ell) \) if \( q_\ell \) is the output produced in period 1. In this Perfect Bayesian equilibrium the more efficient agent continues to earn the informational rent every period.

**Proof:** The claim is that the strategy combination \( \{(\hat{\sigma}^P, \hat{\sigma}^{A\ell})_{\ell=1}^L\} \) described below is a Perfect Bayesian equilibrium.

B (i) In period 1 the principal’s strategy \( \hat{\sigma}^P_1 \) is to offer the menu \( \{(T_\ell, q_\ell)_{\ell=1}^L\} \).

(ii) In period 2, the principal offers \( (T_\ell, q_\ell) \) if in period 1 the agent chose the offer \( (T_\ell, q_\ell) \) from the menu \( \{(T_\ell, q_\ell)_{\ell=1}^L\} \), otherwise offer \( T = \theta_L q \) for any \( q \) the agent produces in each period for the next \( K \) periods.

(iii) If the past history has been \( (T_\ell, q_\ell) \) in every period up to \( t - 1 \) then again offer \( (T_\ell, q_\ell) \) in period \( t \).

(iv) If the principal offers \( (q, T) \neq (T_\ell, q_\ell) \) in any period \( t \geq 2 \) and both the principal and the agent had offered and produced \( (T_\ell, q_\ell) \) in all previous periods, then the agent produces \( q = 0 \) from time \( t + 1 \) onwards for \( K \) periods.
This is a phase I punishment strategy.

(v) If the agent produces \( q \neq q^\ell \) in any period \( t \) and both the principal and the agent had offered and produced \((T^\ell_q, q^\ell)\) in all previous periods, then the principal offers \( T = \theta Lq \) for \( K \) periods after that. This is a phase I punishment for the agent.

(vi) If there are no deviations during a phase I punishment by either the principal or the agent, then after the length of time \( K \), the principal offers \((T^\ell_q + \epsilon, q^\ell)\), such that \( T^\ell_q + \epsilon < S(q^\ell) \) if the principal had been the deviator, and offers \((T^\ell_q - \epsilon, q^\ell)\) such that \( T^\ell_q - \epsilon - \theta q^\ell > 0 \) and if \( \ell = 1, \ldots, L - 1 \), and \((T_L, q_L)\) if \( \ell = L \), if the agent had been the deviator.

(vii) If the agent deviates during a phase I punishment for the principal, then the offer switches to \( T = \theta Lq \) if \( q \) is the output in that period, for a length of time \( K \). If the principal deviates while punishing the agent during a phase I punishment then the offer switches to \( q = 0 \) for the next \( K \) periods. Such a punishment is a phase II punishment.

(viii) After a phase II punishment for the principal, the offer switches to \((q^\ell, T^\ell_q + \epsilon)\) and after a phase II punishment for the agent the offer becomes \((q^\ell, T^\ell_q - \epsilon)\).

(ix) If the principal deviates after a phase II punishment, then the phase I punishment for the principal is played after which the offer becomes \((q^\ell, T^\ell_q + \epsilon)\).

(x) Finally, if the agent deviates after a phase II punishment, then the phase I punishment for the agent is played after which the offer becomes \((q^\ell, T^\ell_q - \epsilon)\).

We note that the strategy described in (i) through (x) is similar to the strategy used for the pooling equilibria of theorem 1. Similar arguments then show that there is a \( \delta_P \) and a \( \delta_A \) such that for all \( \delta \geq \delta_P \), the principal cannot gain by offering a different stream of contracts than the one proposed and for all \( \delta \geq \delta_A \) the agent, whatever be his type, cannot gain by producing an output different from the one meant for the type in the menu.

We now show that the strategy combination \( \tilde{\sigma} = \{(\tilde{\sigma}^P, \tilde{\sigma}^A)^\ell\} \) is a perfect Bayesian equilibrium of the repeated contract game. Consider first the case in which the optimal single-period contract is such that it completely separates types. That is, \( q^\ell \neq q^s \) if \( \ell \neq s \). In this case consider the belief
This is an updated belief system that is consistent with \( \tilde{\sigma} \) as in period 1 the only possible outcome for this strategy is in the set \( \{(q_{\ell}, T_{\ell})\}_{\ell=1}^L \). From the construction of \( \tilde{\sigma} \) it should be clear that if the principal updates beliefs such that \( \nu_\ell = 1 \) for some \( \hat{\ell} \in \{1, \ldots, L\} \) then neither the principal nor the agent, if he is type \( \hat{\ell} \), can gain by deviating from \( \tilde{\sigma} \) after any history \( h_t \). Thus the updated belief system together with the strategy \( \tilde{\sigma} \) is a perfect Bayesian equilibrium of the repeated contract game.

Notice that the principal can infer information about the type of the agent after observing the output level of the agent, and will only offer the contract meant for the type of agent that is consistent with the output produced in period 1. It is interesting to note that the offers are different from the complete information offers, as the more efficient types can expect to receive informational rents in each period, whereas the least efficient type is never required to produce \( q = q^*_L \), where \( q^*_L \) maximizes \( S(q) - \theta_L q \), but only \( q^{SB} \) from period 2 onwards. The reason for this is that if the agent suspects that the principal will renege on the implicit arrangement of not paying the informational rent then in period 1 the agent will never produce anything but \( \bar{q}^{SB} \). Thereafter, the agent will always react to attempts by the principal to use the information fully by reverting to a punishment phase of not producing any output for a number of periods. Even in the case of the least efficient agent, the agent may not want to produce any more than \( q^{SB} \) as the agent has nothing to gain. The principal may in some cases prefer this as the informational rent that has to be given to the agent is in some cases less than the informational rent that has to be given if the complete-information first-best contract is to be implemented.

We next examine the case in which there is Bunching, that is, the same contract is offered to several distinct types. Let \( S_\ell \) denote the set of types that are bunched together with type \( \ell \), that is \( S_\ell = \{s \mid (q_s, T_s) = (q_\ell, T_\ell)\} \). Define the following belief system belief system.

\[
\nu_s|_{h_t, \tilde{\sigma}} = \frac{\nu_s}{\sum_{\lambda \in S_\ell} \nu_\lambda} \quad \text{if } h_1 = (T_s, q_s) \text{ and } s \in S_\ell, \text{ and } \nu_n|_{h_t, \tilde{\sigma}} = 0 \text{ if } n \notin S_\ell.
\]
Thus it is possible that the Principal may offer a sequence of second-best contracts until types have been fully separated. It could be that the perfect Bayesian equilibrium is one in which the $\tilde{\sigma}$ after any history $h_t$. Thus, this belief system together with the strategy $\tilde{\sigma}$ is a perfect Bayesian Equilibrium of the repeated contract game.

8 Separating and Efficient Contracts

Having examined both the nature of pooling contracts as well as separating contracts, we now investigate whether it is possible to find a perfect Bayesian equilibrium in which the complete information efficient contract is offered at some point. As before we will look for contracts that are stationary over long periods. We reconsider again the menu of the optimal single-period contract given by $\{T_\ell, q_\ell\}_{\ell=1}^L$. We know that this single-period optimal contract is incentive compatible and may partially separate types but there could be bunching in the sense that the same offer is made to several different types. Let $S_n = \{s : T_s = T_n \text{ and } q_s = q_n\}$ denote the types that are made the same offer as type $n$. In the case of repeated contracts when there is bunching and an offer $(T_\ell, q_\ell)$ is taken by the agent, the updated belief of the principal is that the agent’s type is in $S_n$. If the belief of the principal about the type of the agent is given by $\nu_\ell = \nu_\ell^{L}$, then the updated belief of the principal after observing output level $q_s$ is that the type of the agent is among those in $S_s$ the set of types that will take the offer $(T_s, q_s)$. The probability distribution that then gives the updated belief of the principal is

$$\nu_s^1 = \frac{\nu_s}{\sum_{k \in S_s} \nu_k} \text{ and } \nu_\ell^1 = 0 \text{ if } \ell \notin S_s.$$ 

In this case the optimal single-period contract that the principal can offer the agent is the menu that solves the following problem

$$\text{maximize } \sum_{\ell=1}^L \nu_\ell^1(S(q_\ell) - T_\ell)$$

such that

$$T_\ell - \theta_\ell q_\ell \geq T_{\ell'} - \theta_{\ell'} \text{ for all } \ell' \neq \ell, \text{ and}$$

$$T_\ell - \theta_\ell q_\ell \geq 0 \text{ for all } \ell = 1, \ldots, L.$$
Let \((T^1_\ell, q^1_\ell)\) denote the menu that solves the principal’s problem given above. Again it is possible that the offer of several types may be bunched together, in which case, in the following round, the principal after updating his belief will offer a menu of contracts that maximizes the principal’s expected payoff given the updated beliefs, subject to the incentive constraints and the participation constraints. This is the optimal contract of the principal after a second round of updating beliefs and we will denote this contract as \((T^2_\ell, q^2_\ell)\). In general the optimal contract of the principal will be denoted by \((T^m_\ell, q^m_\ell)\) after \(k\) rounds of updating of the principal’s belief. As the principal continues to update his belief it will lead to a full separation of types after at most \(M\) rounds. What we show in the next result is that there is a perfect Bayesian equilibrium in the repeated contract game in which after \(K\) periods, only the complete information efficient contract is offered. Let \(\{T^*_\ell, q^*_\ell\}_{\ell=1}^L\) denote the complete information efficient contract for the \(L\) different types. Then we have the following.

**Theorem 4 (Full revelation of Type)** In the infinite horizon repeated contract game there is a perfect Bayesian equilibrium in which the complete information efficient contract \((T^*_s, q^*_s)\) is offered after at most \(M\) periods to type \(s\), if \(s\) is the type of the agent and the principal’s updated belief after \(K\) periods is \(\nu^M_\ell = 1\) if \(\ell = s\) and \(\nu^M_\ell = 0\) otherwise. In this perfect Bayesian equilibrium, the agent of type \(\ell \neq L\) is offered the discounted sum of his informational rent in period 1. If the output in period 1 is consistent with the principal’s updated belief that the type of the agent is bunched with other types, then in period 2 when offering the contract \((T^1_\ell, q^1_\ell)\), the principal offers the discounted sum of the information rents from period 2 onwards to those types in the bunch that are given information rents. This payment of the discounted sum of the information rents continue until there is no further bunching.

**Proof:** We claim that the strategy combination \((\sigma^P, \sigma^A)\) described below is a perfect Bayesian equilibrium in which after at most \(M\)-periods the complete information efficient contract is offered.

(i)In period 1 the principal offers the menu \(\sigma^P_1 = \{T_\ell, q_\ell\}_{\ell=1}^L\), and for the next \(M\) periods offers the contracts \((T^m_\ell, q^m_\ell)\), unless the agent produces an output level that is different from the output levels listed in the menu for that period.
If that happens then the agent is considered to have deviated from the agent’s strategy of producing one of the output levels prescribed by the menu.

(ii) The agent chooses to produce an output level that is among the menu offered by the principal in that period.

(iii) If the principal or the agent deviates from this strategy in any period for the first time, then the phase I punishments as described in Theorem 1 and Theorem 2 is used for the next $K$ periods. If there no deviations during the punishment phase, then the rest of the infinite-horizon game is played with the deviator getting $\epsilon$ less and the non-deviator getting $\epsilon$ more from a choice in the menu that was offered at the time the deviation took place.

(iv) If either the principal or the agent deviates from a phase I punishment then the punishment phase is switched so the deviator is punished and the game switches to the contract that was offered at the time the first deviation took place with the deviator getting $\epsilon$ less and the non-deviator getting $\epsilon$ more. The agent then chooses one of the output levels in the menu offered by the principal.

(v) Finally, if there is a deviation from the stage where a former deviator is getting $\epsilon$ less then the game proceeds to a phase I punishment again.

We first note that as the types with lower marginal cost are given the discounted sum of their information rents, there is no incentive for the agent if he is any of these types to produce an output level inconsistent with its type. This is true at every stage in which the offer of the single-period optimal contract leads to bunching.

Further using the methods outlined in the proof of Theorem 2, it can be shown that, given the updated belief of the principal, at any stage of infinite-horizon repeated contract game, neither player can make profitable gains by deviating from the strategy. Hence, it is a perfect Bayesian equilibrium of the infinite-horizon repeated contract game. It also follows that after at most $M$ periods the principal learns the actual type of the agent.

Theorem 5 (Optimality) The Perfect Bayesian equilibrium that gives the highest expected profit to the principal is the separating perfect Bayesian equilibrium in which the principal offers the single-period second-best optimal contract in period 1 and from period 2 onwards the agent of type $\theta_{\ell}$ is paid $T_{\ell}$ and
asked to produce \( q_\ell \) in every period. That is, the agent is paid in each period what he would have received in the second-best optimal contract. The updated belief of the principal about the type of the agent is consistent with the type of the agent revealed in period 1.

**Proof:** From the proof of lemma 2 we have that

\[
\sum_{\ell=1}^{L} \nu_{\ell}\bar{U}_P(q_1^*, q_2^{SB}, \cdots, q_L^{SB}) \geq \sum_{\ell=1}^{L} \nu_{\ell}\bar{U}_P(q_1^*, q_2^*, \cdots, q_L^*) \\
\geq \sum_{\ell=1}^{L} \nu_{\ell}\bar{U}_P(q_L^*, q_L^*, \cdots, q_L^*).
\]

These expected payoffs include the informational rents that have to paid in order for the agent to choose an output level consistent with its type, and thus show that the discounted payoff from the separating contract, in which the second-best optimal contract is offered in every period, is the one that is optimal for the principal.

\[\blacksquare\]

9 Conclusion

The results here indicate the nature of contracts that one would expect to see in situations that involve repeated interactions between a principal and an agent. We see that learning can take place, and if it does, it happens quickly as the agent can be induced to reveal his type as long as the principal does not take too much advantage of this information. One of the more interesting feature about these long run contracts is the sharp difference that exists between the finite-horizon contracts and the infinite-horizon contracts. This is primarily because there is no credible way in which the principal can offer not to take full advantage of the information that would be revealed by the agent. Thus, the full information rent of the agent has to be paid up front in period 1 itself. In the case of the infinite-horizon the information rent payment can be phased out over the long run as there is a credible way in which the principal can be punished if he reneges on the implicit understanding of the nature of the contract in the perfect Bayesian equilibrium.

One might ask as to why information about the type of the agent is not revealed more slowly over time. The answer perhaps is that with discounting
there is no advantage for the principal to any delays in the revelation of the information and neither for the agent as the agent is not unduly punished for revealing the information. It is interesting to observe that if there is no mechanism to deter the principal from taking full advantage of the information revealed by the agent then the agent would be much less willing to reveal his type and the resulting output less than optimal for the principal.

References


