Consistency, converse consistency and strategic justifications in the airport problem

Cheng-Cheng Hu† Min-Hung Tsay‡ Chun-Hsien Yeh§

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Abstract
We study the “airport problem”, which is concerned with sharing the cost of an airstrip among agents who need airstrips of different lengths. We address whether different formulations of consistency and converse consistency axioms provide strategic justifications for different rules. We propose Right-endpoint Subtraction (RS) bilateral consistency and RS converse consistency in the airport problem, and find that the nucleolus satisfies the two properties. We then introduce a 3-stage extensive form game to implement the nucleolus that exploits the properties. As we show, there is a unique subgame perfect equilibrium outcome of the game and moreover, it is the allocation chosen by the nucleolus. Our result together with Hu et al. (2011)’s strategic justification of the constrained equal benefits rule provides a positive answer to the question. Journal of Economic Literature Classification Numbers: C71; C72; D63; D70.

Keywords: Bilateral consistency; converse consistency; subgame perfect equilibrium; nucleolus; airport problem

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†Department of Economics, National Cheng Kung University, Tainan 701, Taiwan. E-mail: hucc@mail.ncku.edu.tw

‡Department of Finance, National Central University, Chungli 320, Taiwan. E-mail: mhtsay@hotmail.com

§Corresponding author: Institute of Economics, Academia Sinica, Taipei 115, Taiwan. E-mail: chyeh@econ.sinica.edu.tw
1 Introduction

We consider the following class of “airport problems”, which exemplifies the problem of sharing the cost of a facility among agents who have different needs for it, but serving a given agent allows serving all agents with smaller needs than hers at no extra cost. An airstrip has to be built to serve a group of airlines. Different airlines need airstrips of different lengths. The larger a plane, the longer the airstrip it needs. An airstrip that accommodates a given plane can accommodate all smaller planes than its at no extra cost. To accommodate all planes, the airstrip must be long enough for the largest plane. How should the cost of the airstrip for the largest plane be shared among the airlines with different costs of the airstrips? A “rule” is a function that associates with each airport problem, an allocation of the cost of the airstrip. We call this allocation a “contributions vector”.

Consistency and converse consistency principles have been applied to a number of models such as taxation, bargaining, social choice etc. It is well-known that different formulations of consistency and converse consistency axioms provide axiomatic justifications for different rules. However, little attention has been paid to whether the same phenomenon occurs from the strategic perspective. The objective of our paper is to offer a positive answer.

It is well-known that different formulations of a reduced problem lead to different consistency and converse consistency axioms. There are two central formulations, left-endpoint formulation and right-endpoint formulation.
in the airport problem. Hu et al. (2011) adopt the former to propose “Left-endpoint Subtraction (LS) bilateral consistency” and “LS converse consistency”, and then introduce an extensive form game to implement the “Constrained Equal Benefits (CEB) rule” (Potters and Sudhölter, 1999)\(^5\) that exploits the properties.

To serve our purpose, we adopt the right-endpoint formulation to propose “Right-endpoint Subtraction (RS) bilateral consistency” and “RS converse consistency”, and take Hu et al. (2011)’s game form as a blueprint to revise their game form in a way that RS bilateral consistency and RS converse consistency are taken into account. As we show, the “nucleolus” (Schmeidler, 1969),\(^6\) which lexicographically maximizes the “welfare” of the worst-off coalitions, is RS bilaterally consistent and RS conversely consistent and moreover, it is implemented by the following game form.

**Stage 1:** Each agent, except for one of the agents with the largest need called the responder announces a number interpreted as a contribution to the total cost (the cost of satisfying an agent with the largest need).

**Stage 2:** The responder takes one of the following actions. She makes all other agents contribute the amounts they specified, whereas she contributes the residual cost. Alternatively, she takes one agent as her partner to the next stage. All other agents contribute the amounts they specified.

**Stage 3:** The partner chooses helpers from all other agents. The responder and her partner contribute expected amounts specified as follows. Imaging that there is a fair coin to select one of the two agents. The chosen agent is

\(^5\)The rule equalizes agents’ benefits (at a contributions vector, the benefits of an agent is the difference between the cost of satisfying her need and her contribution) subject to no one receiving a subsidy. Potters and Sudhölter (1999) refer to it as the “modified nucleolus”. We adopt Thomson (2004)’s terminology.

\(^6\)The nucleolus is a direct application of the nucleolus, which lexicographically maximizes the “welfare” of the worst-off coalitions, a central solution in Transferable Utility (TU) games. Any solution defined on the class of TU games can be used to provide recommendations for airport problems. There are several ways of transforming an airport problem into an associated TU game. One way to do it is to define the worth of a coalition as the largest cost of any member in that coalition. We then apply a solution in TU games to solve the associated TU game. This yields a payoff vector. We take this payoff vector as the contributions vector for the airport problem.
given priority to choose helpers from all other agents and use their contributions to build the part of the airstrip that she and her helpers can use. The other agent then uses the leftover\(^7\) to cover the difference between the total cost and the cost of the part of the airstrip already built.

Hu et al. (2011) adopt the left-endpoint formulation to define “Left-endpoint Subtraction (LS) bilateral consistency” and “LS converse consistency”. By exploiting the two properties,\(^8\) the authors introduce an extensive form game to implement the “Constrained Equal Benefits (CEB) rule”\(^9\) (Potters and Sudhölter, 1999). The authors show that there is a unique Subgame Perfect Equilibrium (SPE) outcome of the game, which is the allocation chosen by the CEB rule.

Our non-cooperative foundations for the nucleolus and Hu et al. (2011)’s non-cooperative foundations for the CEB rule show that different formulations of a reduced problem provide justifications for the two rules from the strategic perspective.

This paper is organized as follows. Section 2 introduces the model, and central rules and properties. Section 3 develops the axiomatic viewpoint. Section 4 provides the strategic viewpoint.

2 Preliminary

Let \( U \subseteq \mathbb{N} \) be a universe of agents with at least two elements, where \( \mathbb{N} \) is the set of natural numbers. An airport problem, or simply a problem, is a pair \( (N, c) \) where \( N \subseteq U \) is a finite nonvoid agent set and \( c \equiv (c_i)_{i \in N} \in \mathbb{R}^N_+ \) is the profile of agents’ costs. Let \( \mathcal{A} \) be the class of all problems on \( U \). A contributions vector of \( (N, c) \in \mathcal{A} \) is a vector \( x \in \mathbb{R}^N \) satisfying two conditions. First is “efficiency”: the sum of all contributions should be equal to

\(^7\)By leftover, we mean the difference between the sum of the contributions already made and the cost of the part of the airstrip already built if positive; otherwise, zero.

\(^8\)Krishna and Serrano (1996) suggest to use the properties of a rule as guides in designing a game that implements the rule. For this line of research, see also Serrano (1997), Dagan et al. (1997), Serrano and Vohra (2002), Chang and Hu (2008), Chun et al. (2011), and others. For a survey on the Nash program, see Serrano (2005).

\(^9\)The CEB rule equalizes agents’ benefits (at a contributions vector, the benefits of an agent is the difference between her cost and her contribution) subject to no one receiving a subsidy. Potters and Sudhölter (1999) refer to it as the “modified nucleolus”. We adopt Thomson (2004)’s terminology.
the total cost. Formally, $\sum_{i \in N} x_i = \max_{i \in N} c_i$. Second is “reasonableness”: each agent should not receive a subsidy and should not contribute more than her cost. Formally, for each $i \in N$, $0 \leq x_i \leq c_i$. Let $X(N, c)$ be the set of all contributions vectors for $(N, c) \in \mathcal{A}$. A rule is a function defined on $\mathcal{A}$ that associates with each problem $(N, c) \in \mathcal{A}$ a vector $x \in X(N, c)$. Let $n \equiv |N|$ and for simplicity, assume that $N \equiv \{1, \ldots, n\}$ and $c_1 \leq \cdots \leq c_n$. Thus, the agents are ordered in terms of their costs. We refer to agent 1 (the agent with the smallest cost and the lowest index) as the first agent and to agent $n$ (the agent with the largest cost and the highest index) as the last agent. Our generic notation for a rule is $\varphi$. For each $N' \subset N$, we denote $(c_i)_{i \in N'}$ by $c_{N'}$, $(\varphi_i(N, c))_{i \in N'}$ by $\varphi_{N'}(N, c)$, and so on.

We formally introduce the nucleolus and our central properties. Since for general games, the payoff vector the nucleolus chooses is obtained by solving a sequence of linear programs, it is in general not easy to compute. However, for airport problems, the nucleolus can be calculated by an explicit formula (Littlechild, 1974; Sönmez, 1994). For our purpose, Sönmez’s formula is the most convenient.

**Nucleolus, Nu:** For each $(N, c) \in \mathcal{A}$,

$$
Nu_1(N, c) \equiv \min_{1 \leq k \leq n-1} \left\{ \frac{c_k}{k+1} \right\}
$$

$$
Nu_i(N, c) \equiv \min_{i \leq k \leq n-1} \left\{ \frac{c_k - \sum_{p=1}^{i-1} Nu_p(N, c)}{k+i+2} \right\} \quad \text{where } 2 \leq i \leq n-1
$$

$$
Nu_n(N, c) \equiv c_n - \sum_{p=1}^{n-1} Nu_p(N, c).
$$

**Remark 1:** A number of rules (e.g., the nucleolus and the CEB rule) coincide with the “standard rule”\(^{10}\) for two-agent problems.

To introduce our central properties, consider the three-agent problem in Figure 1 and a contributions vector $x$ chosen by a rule for it. Suppose that one agent, say agent 2, pays her contribution $x_2$ and “leaves”, and reassess

\(^{10}\) The rule is defined for two-agent problems. It says that the agent with the smaller cost contributes half of her cost, and the other contributes the remainder. Formally, for each $((i, j), (c_i, c_j)) \in \mathcal{A}$ with $c_i \leq c_j$, $S_i((i, j), (c_i, c_j)) \equiv \frac{c_i}{2}$ and $S_j((i, j), (c_i, c_j)) \equiv c_j - S_i((i, j), (c_i, c_j))$.  

There are three agents. For simplicity, assume that $c_1 < c_3$. Thus, agent 1 uses segment 1. Agent 2 uses segments 1 and 2. Agent 3 uses segments 1, 2, and 3. The cost of segment 1 is $c_1$. The cost of segment 2 is $c_2 - c_1$. The cost of segment 3 is $c_3 - c_2$. The total cost is $c_3$, which is the sum of $c_1$ (the cost of segment 1), $c_2 - c_1$ (the cost of segment 2), and $c_3 - c_2$ (the cost of segment 3).

Figure 1: Three-agent problem

the situation from the viewpoint of the remaining agents. The remaining agents are now left with the amount $x_2$ that can help cover part of the total cost $c_3$. Instead of thinking of $x_2$ as covering an abstract part of the airstrip, it is natural to think of $x_2$ as intended to help cover the segments agent 2 uses (namely, segments 1 and 2). The question is which segmental cost should be covered first. Our central properties are based on the right-endpoint formulation, which suggests to first cover the cost of segment 2 (the segment agent 2 uses but agent 1 does not); unless $x_2 > c_2 - c_1$, in which case, $c_2 - c_1$ is completely covered and the remainder ($x_2 - (c_2 - c_1)$) would be used to help cover the cost of segment 1 (the segment agents 2 and 1 use). This implies that if $c_2 - c_1 > x_2$, since $x_2$ is too little to cover the cost of segment 1 (the segment agent 1 uses), agent 1’s cost is revised down by 0; otherwise, $c_2 - c_1$ is completely covered and the remainder ($x_2 - (c_2 - c_1)$) would be used to help cover the cost of segment 1. Thus, agent 1’s cost is revised down by $x_2 - (c_2 - c_1)$ and her revised cost is $c_2 - x_2$. Altogether, agent 1’s revised cost is the minimum of $c_1$ and $c_2 - x_2$. Since contributing to the segments agent 2 uses implies contributing to the segments agent 3.
uses, agent 3’s cost is then revised down by $x_2$.\footnote{The left-endpoint formulation suggests to first cover the cost of segment 1 (the segment all agents use); unless $x_2 > c_1$, in which case, $c_1$ is completely covered and the remainder $(x_2 - c_1)$ would be used to help cover the cost of segment 2 (the segment agents 2 and 3 use). This implies that for each agent, say agent $j$, her cost is revised down by $x_2$. Since no agent’s cost can be a negative, agent $j$’s revised cost is the maximum of $c_j - x_2$ and zero. A rule satisfies “Left-endpoint Subtraction (LS) consistency” (Potters and Sudhölter, 1999) if a contributions vector $x$ is chosen by the rule for a problem, then for the reduced problem just defined, the components of $x$ pertaining to the remaining agents should still be chosen by the rule. Hu et al. (2011) adopt the formulation to define the two-agent reduced problem and propose “LS bilateral consistency” and “LS converse consistency”.}

A rule is “RS consistent” (Potters and Sudhölter, 1999) if a contributions vector $x$ is chosen by the rule for a problem, then for the reduced problem just defined, the components of $x$ pertaining to the remaining agents should still be chosen by the rule.\footnote{Potters and Sudhölter (1999) refer to RS consistency and LS consistency as $\nu$-consistency and $\psi$-consistency, respectively. For an additional reference on RS consistency, see Hwang and Yeh (2010).} We adopt the above idea to define the two-agent reduced problem. Formally, let $(N, c) \in \mathcal{A}$ with $|N| \geq 2$, $i \in N \setminus \{n\}$ and $x \in X(N, c)$. The reduced problem of $(N, c)$ with respect to $N' \equiv \{i, n\}$ and $x$, $(N', r^x_{N'})$, is defined by setting\footnote{The following example show that to avoid the negativity problem, we have to take the maximum between $\min_{i \leq k \neq n} \left\{c_k - \sum_{m \leq k, m \neq i, n} x_m \right\}$ and zero in the definition of the reduced problem. Let $N \equiv \{1, \cdots, 5\}$, $c \equiv (2, 3, 4, 5, 10)$ and $x \equiv (2, 2, 2, 2, 2)$. Let $N' \equiv \{4, 5\}$. Then $(r^x_{N'})_4 \equiv c_4 - x_1 - x_2 - x_3 = -1 < 0$. We next show that to avoid the efficiency problem, the possibility of the last agent leaving is excluded. Let $N \equiv \{1, \cdots, 5\}$, $c \equiv (3, 4, 5, 6, 16)$ and $x \equiv (2, 2, 3, 3, 6)$. Let $N' \equiv \{3, 4\}$. Then $(r^x_{N'})_3 \equiv \min \{c_3 - x_1 - x_2, c_5 - x_1 - x_2 - x_5 \} = 1$ and $(r^x_{N'})_4 \equiv \min \{c_4 - x_1 - x_2, c_5 - x_1 - x_2 - x_5 \} = 2$. Note that $(r^x_{N'})_3 + (r^x_{N'})_4 = 3 < x_3 + x_4 = 6.$}

$$
(r^x_{N'})_i \equiv \max \left\{ \min_{i \leq k \neq n} \left\{ c_k - \sum_{m \leq k, m \neq i, n} x_m \right\}, 0 \right\}, \\
(r^x_{N'})_n \equiv \max \left\{ \min_{n \leq k, k \neq i} \left\{ c_k - \sum_{m \leq k, m \neq i, n} x_m \right\}, 0 \right\}.
$$

A rule is “RS bilaterally consistent” if a contributions vector $x$ is chosen by...
the rule for a problem, then for the two-agent reduced problem just defined, the components of \( x \) pertaining to the remaining agents should still be chosen by the rule.

**Right-endpoint Subtraction (RS) bilateral consistency**: For each \((N, c) \in A\) with \(|N| \geq 2\) and each \(i \in N\{n\}\), if \(x = \varphi(N, c)\), then \((\{i, n\}, r_{\{i,n\}}^x) \in A\) and \(x_{\{i,n\}} = \varphi(\{i, n\}, r_{\{i,n\}}^x)\).

A rule is “RS conversely consistent” if a contributions vector \(x\) is such that for each two-agent reduced problem involving agent \(n\), the restriction of \(x\) to this subgroup is chosen by the rule, then \(x\) should be chosen by the rule for the initial problem.\(^{14}\)

**RS converse consistency**: For each \((N, c) \in A\) with \(|N| > 2\) and each \(x \in X(N, c)\), if for each \(N' \subset N\) with \(|N'| = 2\) and \(n \in N'\), \(x_{N'} = \varphi(N', r_{N'}^x)\), then \(x = \varphi(N, c)\).

Since the nucleolus is RS consistent (Potters and Sudhölter, 1999), the following result is immediate.

**Lemma 1**: The nucleolus is RS bilaterally consistent.

It is well-known that the “prekernel” (Davis and Maschler, 1965) is “max conversely consistent”\(^{15}\) (Peleg, 1986) for TU games. Since the nucleolus coincides with the prekernel for “concave TU games” (Maschler et al., 1972) and airport problems are concave TU games (Littlechild and Owen, 1973), the nucleolus is max conversely consistent in airport problems. Note that RS consistency is an application for airport problems of “max consistency” (Davis and Maschler, 1965) for TU games (Potters and Sudhölter, 1999). A weak version of RS converse consistency, “RS weak converse consistency”, obtained by considering all two-agent subgroups is an application of “max converse consistency” (Peleg, 1986). Thus, the nucleolus satisfies RS weak converse consistency. With the help of this fact, we show the following.

\(^{14}\)It can be shown that RS bilateral consistency and RS converse consistency are logically independent.

\(^{15}\)The terminology is borrowed from Thomson (2010).
Lemma 2: The nucleolus is RS conversely consistent.

Proof. Let \((N, c) \in \mathcal{A}\) and \(x \in X(N, c)\). For each \(k \in N \setminus \{n\}\), let \(\{k, n\}, r_{\{k,n\}}^x\) \(\in \mathcal{A}\) and \(x_{\{k,n\}} = Nu(\{k, n\}, r_{\{k,n\}}^x)\). We show that \(x = Nu(N, c)\). Clearly, for each \(k \in N \setminus \{n\}\), \(\left(r_{\{k,n\}}^x\right)_k \leq \left(r_{\{k,n\}}^x\right)_n = x_k + x_n\).

Since \(Nu(\{k, n\}, r_{\{k,n\}}^x) = S(\{k, n\}, r_{\{k,n\}}^x)\), then \(x_k = \frac{1}{2} \left(r_{\{k,n\}}^x\right)_k\) and \(x_k \leq x_n\). We first show that \(x\) satisfies the hypothesis of RS weak converse consistency, and then by invoking the fact that the nucleolus satisfies weak RS converse consistency, we conclude that \(x = Nu(N, c)\). Let \(i, j \in N \setminus \{n\}\) be such that \(i \neq j\). Without loss of generality, assume that \(x_i \leq x_j\).

Step 1: \(\left(r_{\{i,j\}}^x\right)_j = x_i + x_j\). Suppose, by contradiction, that \(\left(r_{\{i,j\}}^x\right)_j \neq x_i + x_j\). Since for each \(l \in \{i, j\}\), \(\left(r_{\{i,j\}}^x\right)_l \leq c_n - \sum_{m \leq n, m \neq i, j} x_m = x_i + x_j\), then \(\left(r_{\{i,j\}}^x\right)_j < x_i + x_j\). By definition of \(\left(r_{\{i,j\}}^x\right)_j\), there is \(k \in \{j, j + 1, \ldots, n\}\) such that \(k \neq i\) and \(\left(r_{\{i,j\}}^x\right)_j = c_k - \sum_{m \leq k, m \neq i, j} x_m\). Since \(\left(r_{\{i,j\}}^x\right)_j < x_i + x_j\), then \(k \neq n\). By definition of \(\left(r_{\{i,j\}}^x\right)_j\),

\[
\left(r_{\{i,j\}}^x\right)_j \leq c_k - \sum_{m \leq k, m \neq i, j} x_m \leq c_k - \sum_{m \leq k, m \neq i, j} x_m = \left(r_{\{i,j\}}^x\right)_j < x_i + x_j.
\]

Since \(Nu(\{j, n\}, r_{\{j,n\}}^x) = S(\{j, n\}, r_{\{j,n\}}^x)\) and \(x_i \leq x_j\), then \(x_j = \frac{1}{2} \left(r_{\{j,n\}}^x\right)_j < \frac{1}{2} (x_i + x_j) \leq x_j\), which is impossible. Thus, \(\left(r_{\{i,j\}}^x\right)_j = x_i + x_j\).

Step 2: \(\left(r_{\{i,j\}}^x\right)_i = 2x_i\). We first show that \(\left(r_{\{i,j\}}^x\right)_i \leq 2x_i\). By definition of \(\left(r_{\{i,j\}}^x\right)_i\), there is \(k \in \{i, i + 1, \ldots, n\}\) such that \(k \neq n\) and \(\left(r_{\{i,n\}}^x\right)_i = c_k - \sum_{m \leq k, m \neq i, n} x_m\). Since \(Nu(\{i, n\}, r_{\{i,n\}}^x) = S(\{i, n\}, r_{\{i,n\}}^x)\), then \(x_i = \frac{1}{2} \left(r_{\{i,n\}}^x\right)_i\) and \(c_k - \sum_{m \leq k, m \neq i, n} x_m = 2x_i\). If \(k < j\), then \(c_k - \sum_{m \leq k, m \neq i, n} x_m = \)
$c_k - \sum_{m \leq k, m \neq i,j} x_m$, which implies that

$$\left( r^x_{\{i,j\}} \right)_i \leq c_k - \sum_{m \leq k, m \neq i,j} x_m = c_k - \sum_{m \leq k, m \neq i,n} x_m = \left( r^x_{\{i,n\}} \right)_i = 2x_i.$$ 

If $k \geq j$, since $2x_i = \left( r^x_{\{i,n\}} \right)_i = c_k - \sum_{m \leq k, m \neq i,n} x_m$ and

$$2x_j = \left( r^x_{\{i,n\}} \right)_j \leq c_k - \sum_{m \leq k, m \neq j,n} x_m = c_k - \sum_{m \leq k, m \neq i,n} x_m - x_i + x_j = x_i + x_j,$$

then $x_j \leq x_i$. Since $x_i \leq x_j$, $x_i = x_j$. Since $\left( r^x_{\{i,j\}} \right)_i \leq c_n - \sum_{m \leq n, m \neq i,j} x_m = x_i + x_j$, then $\left( r^x_{\{i,j\}} \right)_i \leq 2x_i$. We conclude that $\left( r^x_{\{i,j\}} \right)_i \leq 2x_i$.

We now show that $\left( r^x_{\{i,j\}} \right)_i = 2x_i$. Suppose, by contradiction, that $\left( r^x_{\{i,j\}} \right)_i \neq 2x_i$. Then, $\left( r^x_{\{i,j\}} \right)_i < 2x_i$. By definition of $\left( r^x_{\{i,j\}} \right)_i$, there is $k' \in \{i, i+1, \cdots, n\}$ such that $k' \neq j$ and $\left( r^x_{\{i,j\}} \right)_i = c_{k'} - \sum_{m \leq k', m \neq i,j} x_m < 2x_i$. Since $x_i \leq x_j$, then $k' \neq n$. If $k' < j$, since $c_{k'} - \sum_{m \leq k', m \neq i,n} x_m = c_{k'} - \sum_{m \leq k', m \neq i,n} x_m = Nu \left( \{i, n\}, r^x_{\{i,j\}} \right) = S \left( \{i, n\}, r^x_{\{i,n\}} \right)$, then

$$2x_i = \left( r^x_{\{i,n\}} \right)_i \leq c_{k'} - \sum_{m \leq k', m \neq i,n} x_m = c_{k'} - \sum_{m \leq k', m \neq i,n} x_m = \left( r^x_{\{i,j\}} \right)_i < 2x_i,$$

which is impossible. If $k' > j$, then

$$2x_i = \left( r^x_{\{i,n\}} \right)_i \leq c_{k'} - \sum_{m \leq k', m \neq i,n} x_m = \left( r^x_{\{i,j\}} \right)_i - x_j < 2x_i,$$

which is again impossible. Thus, $\left( r^x_{\{i,j\}} \right)_i = 2x_i$.

Since the nucleolus coincides with the standard rule for two-agent problems, by Steps 1 and 2, $x_{\{i,j\}} = Nu(\{i, j\}, r^x_{\{i,j\}})$, which implies that for each $N' \subseteq N$ with $|N'| = 2$, $x_{N'} = Nu(N', r^x_{N'})$. Since the nucleolus satisfies weak RS converse consistency, $x = Nu(N, c)$. Q.E.D.
Remark 2: Hu et al. (2011) show that the CEB rule is the only LS bilateral consistent (or LS conversely consistent) rule satisfying “equal treatment of equals”\(^{16}\) and “last-agent cost additivity”\(^{17}\). By invoking the Elevator Lemma (Thomson, 2010) and Proposition 2 in Hu et al. (2011), it can be shown that the nucleolus is the only RS bilateral consistent (or RS conversely consistent) rule satisfying equal treatment of equals and last-agent cost additivity. Thus, these results show that different formulations of consistency axiom provide axiomatic justifications for the two rules.

3 A strategic justification

We modify Hu et al. (2011)’s game by revising the contributions of the responder and her partner at the final stage on the basis of the reduced problem underlying the definitions of RS bilateral consistency and RS converse consistency. This is the only difference between the two games.

Let \((N, c) \in \mathcal{A}\) with \(c \equiv (c_i)_{i \in N} \in \mathbb{R}_{++}^N\). Our 3-stage extensive form game is denoted by \(\Omega(N, c)\), and the game tree depicted in Figure 2.

**Stage 1:** Each agent \(k \in N \setminus \{n\}\) independently announces a number \(x_k \in \mathbb{R}\) such that \(0 \leq x_k \leq c_k\). Let \(x_n \equiv c_n - \sum_{k \neq n} x_k\) We refer to \(x \equiv (x_k)_{k \in N}\) as the proposal.

**Stage 2:** Agent \(n\) adopts one of the following two strategies:

1. Accept \(x_n\) (in short, action \(A\)). The game ends with \(x\) as the outcome.

2. Reject \(x_n\) and pick one agent from \(N \setminus \{n\}\), say agent \(i\) (in short, action \((R, i)\)). The game proceeds to the final stage.

**Stage 3:** Agent \(i\) pick one agent from \(\{i, \cdots , n - 1\}\), say agent \(j\) (in short, action \(j\)), to work with. The game ends with \((\tau_i^j(c, x), \tau_n^j(c, x) , x_{N \setminus \{i, n\}})\)

\(^{16}\)Two agents with equal costs should contribute equal amounts.

\(^{17}\)If the last agent’s cost increases by \(\delta\), her contribution should increase by \(\delta\).
Stage 1:
Each agent $k \in N \setminus \{n\}$ announces a number $x_k$ such that $0 < x_k < c_k$. Let $x_n = c_n - \sum_{k \neq n} x_k$.

Stage 2:
Agent $n$ decides to take $A$ (accept $x_n$) or $(R, i)$ (reject $x_n$ and choose one agent from $N \setminus \{n\}$, say agent $i$).

Stage 3:
Agent $i$ chooses one agent from $\{i, \ldots, n-1\}$, say agent $j$.

$\begin{align*}
 & (x_1, \ldots, x_{i-1}, \tau^i(c, x), x_{i+1}, \ldots, x_{n-1}, \tau^n(c, x))
\end{align*}$

Figure 2: The game tree of $\Omega(N, c)$
as the outcome, where each agent $k \in \mathbb{N} \backslash \{i, n\}$ contributes $x_k$, and agents $i$ and $n$ contribute

$$
\tau^i_i(c, x) \equiv \frac{1}{2} \left\{ c_j - \sum_{m \leq j, m \neq i, n} x_m + x_i + x_n - \left( c_n - \sum_{m \leq n, m \neq i, n} x_m \right) \right\}
$$

$$
\tau^i_n(c, x) \equiv \frac{1}{2} \left\{ c_n - \sum_{m \leq n, m \neq i, n} x_m + x_i + x_n - \left( c_j - \sum_{m \leq j, m \neq i, n} x_m \right) \right\}
$$

We next elaborate $\Omega(N, c)$ and discuss the differences between our game and Hu et al. (2011)’s game.

1. In Stage 3, the contributions of agent $n$ and her partner, say agent $i$, are constructed on the basis of the reduced problem underlying RS bilateral consistency and RS converse consistency. We can think of the construction as made by imagining that the agent chosen by Nature is given priority to select one agent whose cost is not less than her cost, say agent $k$, and use $\sum_{m \leq k, m \neq i, n} x_m$ to cover $c_k$; the other agent uses the remainder ($\sum_{m \leq n, m \neq i, n} x_m - \sum_{m \leq k, m \neq i, n} x_m$) to cover the residual cost ($c_n - c_k$).

2. In contrast to Hu et al. (2011)’s game, the only difference is the construction of the agents’ contributions in Stage 3. In their game, the agent chosen by Nature has no choice but uses $\sum_{m \leq n, m \neq i, n} x_m$ to cover her cost. However, in our game, she can select one agent whose cost is not less than her cost, say agent $k$, and use $\sum_{m \leq k, m \neq i, n} x_m$ to cover $c_k$.

The following notation is used to construct a SPE of $\Omega(N, c)$. Let $x \in \mathbb{R}^N$ and $i \in \mathbb{N} \backslash \{n\}$. When $x$ is the proposal, agent $n$ takes agent $i$ in Stage 2, and agent $i$ takes agent $k$ in Stage 3, the contributions of agents $i$ and $n$ in Stage 3, denoted respectively by $\tau^i_i(c, x, k)$ and $\tau^i_n(c, x, k)$, are defined as follows:

$$
\tau^i_i(c, x, k) \equiv \frac{1}{2} \left( c_k - \sum_{m \leq k, m \neq i} x_m \right)
$$
and

\[ \tau_n^i (c, x, k) \equiv x_i + x_n - \tau_i^i (c, x, k). \]

Let \( \tau^i (c, x, k) \equiv (\tau_i^i (c, x, k), \tau_n^i (c, x, k)) \) be the contributions vector of agents \( i \) and \( n \) in Stage 3. The expected contributions of agents \( i \) and \( n \) in Stage 2, denoted respectively by \( \tau_i^i (c, x) \) and \( \tau_i^n (c, x) \), are defined as follows:

\[ \tau_i^i (c, x) \equiv \min_{i \leq k < n} \tau_i^i (c, x, k) \]

and

\[ \tau_i^n (c, x) \equiv x_i + x_n - \tau_i^i (c, x). \]

Let \( \tau^i (c, x) \equiv (\tau_i^i (c, x), \tau_i^n (c, x)) \) be the expected contributions vector of agents \( i \) and \( n \) in Stage 2.

The following two results establish a strategic justification for the nucleolus. First, the allocation chosen by the nucleolus can be supported as a SPE of \( \Omega(N, c) \). Second, the SPE outcome of \( \Omega(N, c) \) is unique and moreover, it is the allocation chosen by the nucleolus. As usual, we solve \( \Omega(N, c) \) by backward induction.

**Theorem 1:** Let \((N, c) \in A\) with \( c \equiv (c_i)_{i \in N} \in \mathbb{R}^N_+\). There is a SPE of \( \Omega(N, c) \) with outcome \( Nu(N, c) \).

**Proof.** Let \((N, c) \in A\) with \( c \equiv (c_i)_{i \in N} \in \mathbb{R}^N_+\). The proof is by construction of a strategy profile \( f \) that generates \( Nu(N, c) \) as the outcome. The profile \( f \) is defined as follows.

**Stage 1:** Each agent \( k \in N \setminus \{n\} \) announces \( Nu_k(N, c) \).

**Stage 2:** Let \( x \) be the proposal. Let \( \mu \equiv \min_{i \in N \setminus \{n\}} \tau_n^i (c, x) \). If \( x_n \leq \mu \), agent \( n \) takes action \( A \); otherwise, she takes action \( R \) and chooses one agent from \( N \setminus \{n\} \), say agent \( i \), such that \( \tau_n^i (c, x) = \mu \).

**Stage 3:** Agent \( i \) chooses one agent from \( \{i, \ldots, n - 1\} \), say agent \( k \), such that \( \tau_i^j (c, x, k) = \min_{i \leq j < n} \tau_i^j (c, x, j) \).
We first show that following \( f \), the outcome is the allocation chosen by the nucleolus, and then conclude the proof by showing that \( f \) is a SPE.

**Step 1: Following \( f \), the outcome is \( Nu(N, c) \).** Following \( f_k \), each agent \( k \in N \setminus \{n\} \) announces \( Nu_k(N, c) \) in Stage 1. Let \( \bar{x} \equiv Nu(N, c) \). Then \( \bar{x} \) is the proposal. Since the nucleolus is \( RS \) bilaterally consistent and coincides with the standard rule for two-agent problems, then for each \( i \in N \setminus \{n\} \),

\[
\tau_i^i(c, \bar{x}) = \frac{1}{2} \min_{i \leq k < n} \left( c_k - \sum_{m \leq k, m \neq i} \bar{x}_m \right) = \frac{1}{2} \left( \tau_{i,n}^i(c) \right)_i = \bar{x}_i
\]

and

\[
\tau_n^i(c, \bar{x}) = \bar{x}_n,
\]

which implies that \( \bar{x}_n = \mu \). Following \( f_n \), agent \( n \) takes action \( A \) and the game ends with outcome \( Nu(N, c) \).

**Step 2: \( f \) is a SPE.** Let \( i \in N \setminus \{n\} \). It is easy to see that following \( f_i \) in Stage 3 and \( f_n \) in Stage 2 are best responses for agents \( i \) and \( n \). We show that following \( f_i \) in Stage 1 is a best response for agent \( i \). Let \( \epsilon \in \mathbb{R} \) be such that \( 0 < \epsilon \leq \bar{x}_i \). Suppose that agent \( i \) deviates in Stage 1 by announcing \( x'_i = \bar{x}_i - \epsilon \). Let \( x' \) be the proposal after agent \( i \)'s deviation. Then \( x'_n = \bar{x}_n + \epsilon \), and for each \( j \in N \setminus \{i, n\} \), \( x'_j = \bar{x}_j \). Since \( \bar{x}_n = \tau_n^i(c, \bar{x}) \) and for each \( l \in N \setminus \{n\} \), \( \bar{x}_l = \tau_l^i(c, \bar{x}) \), then for each \( j \in N \setminus \{i, n\} \),

\[
\tau_j^i(c, x') = \min_{j \leq k < n} \tau_j^i(c, x', k) = \frac{1}{2} \min_{j \leq k < n} \left( c_k - \sum_{m \leq k, m \neq j} x'_m \right) = \frac{1}{2} \min_{j \leq k < n} \left( c_k - \sum_{m \leq k, m \neq j} \bar{x}_m + \epsilon \right) = \tau_j^i(c, \bar{x}) + \frac{\epsilon}{2}
\]
and

\[ \tau^i_n(c, x') = x'_j + x'_n - \tau^i_j(c, x') \]

\[ \geq \bar{x}_j + \bar{x}_n + \varepsilon - \left\{ \tau^i_j(c, \bar{x}) + \frac{1}{2}\varepsilon \right\} \]

\[ = \bar{x}_n + \frac{1}{2}\varepsilon \]

\[ > \bar{x}_n \]

\[ = \tau^i_n(c, \bar{x}). \]

Following \( f_n \), agent \( n \) takes agent \( i \) in Stage 2. Since agent \( i \)'s contribution is \( \bar{x}_i \), she is not better off deviating. Let \( \varepsilon \in \mathbb{R} \) be such that \( 0 < \varepsilon \leq c_i - \bar{x}_i \).

Suppose that agent \( i \) deviates in Stage 1 by announcing \( x'_i = \bar{x}_i + \varepsilon \). Then, it can be shown that following \( f_n \), agent \( n \) takes action \( A \). Since agent \( i \)'s contribution is \( x'_i > \bar{x}_i \), she is not better off deviating. \( Q.E.D. \)

**Theorem 2:** Let \( (N, c) \in \mathcal{A} \) with \( c \equiv (c_i)_{i \in N} \in \mathbb{R}_+^N \). Each SPE outcome of \( \Omega(N, c) \) is \( Nu(N, c) \).

**Proof.** Let \((N, c) \in \mathcal{A} \) with \( c \equiv (c_i)_{i \in N} \in \mathbb{R}_+^N \) and \( g \) be a SPE of \( \Omega(N, c) \). Following \( g_k \), each agent \( k \in N \setminus \{n\} \) announces \( x_k \) with \( 0 \leq x_k \leq c_k \) in Stage 1. Let \( x_n \equiv c_n - \sum_{k \neq n} x_k \) and \( x \equiv (x_i)_{i \in N} \) be the proposal. In Stage 2, following \( g_n \), agent \( n \) takes either action \( A \) or action \( R \).

**Case 1: Agent \( n \) takes action \( A \).** The game ends with outcome \( x \). We show that \( x = Nu(N, c) \). We first prove the following claim.

**Claim 1:** For each \( i \in N \setminus \{n\} \), \( x_{\{i,n\}} = \tau^i(c, x) \). The proof of the claim is by the following four facts.

**Fact 1:** For each \( i \in N \setminus \{n\} \), if \( x_i > 0 \), then \( x_{\{i,n\}} = \tau^i(c, x) \). Let \( i \in N \setminus \{n\} \) and \( x_i > 0 \). Suppose, by contradiction, that \( x_{\{i,n\}} \neq \tau^i(c, x) \). Since \( \tau^i_1(c, x) + \tau^i_n(c, x) = x_i + x_n \) and in Stage 2, by subgame perfection, \( x_n \leq \tau^i_n(c, x) \). If \( x_n = \tau^i_n(c, x) \), then \( x_{\{i,n\}} = \tau^i(c, x) \), which violates \( x_{\{i,n\}} \neq \tau^i(c, x) \). Thus, \( x_n < \tau^i_n(c, x) \). Since \( \tau^i_1(c, x) + \tau^i_n(c, x) = x_i + x_n \), \( \tau^i_1(c, x) < x_i \). Let \( \varepsilon \in \mathbb{R} \) be such that \( 0 < \varepsilon < \min \{ \tau^i_n(c, x) - x_n, x_i \} \). We show that agent \( i \) is better off announcing \( x_i - \varepsilon \) in Stage 1. Let \( x' \) be the proposal after agent \( i \)'s deviation. Then, \( x'_i = x_i - \varepsilon \), \( x'_n = x_n + \varepsilon \), and for each
For each \( j \) such that \( j > 0 \), we next show that for each \( i < k < n \), there is \( c \) such that \( \tau^i(c,x') = \frac{1}{2} \min_{i \leq k < n} \left( c_k - \sum_{m \leq k, m \neq i} x'_m \right) \). Let \( i \in N \setminus \{ n \} \) and \( x_i > 0 \). By Fact 1, \( x_i = \tau^i(c,x) \).

Fact 2: For each \( i \in N \setminus \{ n \} \), if \( x_i > 0 \), then for each \( k \in \{ i, \ldots, n - 1 \} \), \( c_k = \sum_{m \leq k} x_m > 0 \). Let \( i \in N \setminus \{ n \} \) and \( x_i > 0 \). By Fact 1, \( x_i = \tau^i(c,x) \).

Since \( \tau^i(c,x) = \frac{1}{2} \min_{i \leq k < n} \left( c_k - \sum_{m \leq k, m \neq i} x_m \right) \), then for each \( i \leq k < n \),

\[
\frac{1}{2} \left( c_k - \sum_{m \leq k, m \neq i} x_m \right) \geq x_i,
\]

which implies that \( c_k - \sum_{m \leq k} x_m \geq x_i > 0 \).

Fact 3: For each \( i \in N \setminus \{ n \} \), \( x_i > 0 \). Suppose, by contradiction, that there is \( i \in N \setminus \{ n \} \) such that \( x_i = 0 \). We first show that \( c_i - \sum_{m \leq i, m \neq i} x_m > 0 \).

We consider two possibilities.

- For each \( l < i \), \( x_l = 0 \). Since \( c_i > 0 \), we have \( c_i - \sum_{m \leq i, m \neq i} x_m > 0 \).
- There is \( l < i \) such that \( x_l > 0 \). Let \( j \) be the largest integer in \( \{ 1, \ldots, i \} \) such that \( x_j > 0 \). By Fact 2, \( 0 < c_j - \sum_{m \leq j} x_m = c_j - \sum_{m \leq i, m \neq i} x_m \).

We next show that for each \( i < k < n \), \( c_k - \sum_{m \leq k, m \neq i} x_m > 0 \). We consider two possibilities.

- For each \( l > i \), \( x_l = 0 \). Since \( c_i - \sum_{m \leq i, m \neq i} x_m > 0 \) and \( c_{i+1} \geq c_i \),

\[
c_k - \sum_{m \leq k, m \neq i} x_m \geq c_i - \sum_{m \leq i, m \neq i} x_m > 0.
\]

For each \( j \leq k \leq n - 1 \), since \( x_i = 0 \), then by Fact 2,

\[
c_k - \sum_{m \leq k, m \neq i} x_m = c_k - \sum_{m \leq k} x_m > 0.
\]
Altogether, \( \tau^i_n(c, x) \equiv \frac{1}{2} \min_{i \leq k < n} \left( c_k - \sum_{m \leq k, m \neq i} x_m \right) > 0 \). In Stage 2, by subgame perfection, \( x_n \leq \tau^i_n(c, x) \). Since \( x_i + x_n = \tau^i_1(c, x) + \tau^i_n(c, x) \), then \( \tau^i_i(c, x) \leq x_i = 0 \), which violates that \( \tau^i_i(c, x) > 0 \).

**Fact 4:** \( x \in X(N, c) \). By Facts 1 and 3, for each \( i \in N \setminus \{n\} \), \( x_{\{i,n\}} = \tau^i(c, x) \). By Facts 2 and 3, \( c_{n-1} - \sum_{m \leq n-1, m \neq n} x_m > 0 \), which implies that \( c_{n-1} - \sum_{m \leq n-1, m \neq n-1, n} x_m > x_{n-1} > 0 \). By the definition of \( \tau^{n-1}_{n-1}(c, x) \),

\[
\tau^{n-1}_{n-1}(c, x) = \frac{1}{2} \left( c_{n-1} - \sum_{m \leq n-1, m \neq n-1, n} x_m \right) > 0
\]

and

\[
x_n = \tau^{n-1}_n(c, x) = x_{n-1} + x_n - \tau^{n-1}_{n-1}(c, x)
\]

\[
= \left( c_n - \sum_{m \leq n-1, m \neq n-1, n} x_m \right) - \tau^{n-1}_{n-1}(c, x)
\]

\[
\geq 2\tau^{n-1}_{n-1}(c, x) - \tau^{n-1}_{n-1}(c, x)
\]

\[
= \tau^{n-1}_{n-1}(c, x)
\]

\[
> 0.
\]

By Fact 3, for each \( i \in N \), \( x_i > 0 \). Since \( \sum_{i \in N} x_i = c_n \), \( x \in X(N, c) \).

By Facts 1 and 3, we conclude that the proof of the claim.

By Facts 2 and 3, \( (r^x_{\{i,n\}}) = \min_{i \leq k < n} \left( c_k - \sum_{m \leq k, m \neq i} x_m \right) \), which implies that \( \tau^i(c, x) = Nu(\{i, n\}, r^x_{\{i,n\}}) \). By Fact 4 and RS converse consistency of the nucleolus, \( x = Nu(N, c) \).

**Case 2:** Agent \( n \) takes action \( R \). Suppose that agent \( n \) chooses agent \( i \) in Stage 2. Let \( y \equiv (\tau^i(c, x), x_{N \setminus \{i, n\}}) \) be the outcome. We show that \( y = Nu(N, c) \). If \( |N| = 2 \), since \( N \equiv \{i, n\} \), \( \tau^i_i(c, x) = \tau^i_1(c, x, i) = \frac{c_n}{2} \) and \( \tau^i_n(c, x) = x_i + x_n - \tau^i_1(c, x) = c_n - \frac{c_n}{2} \), which implies that \( y = Nu(N, c) \). Suppose that \( |N| \geq 3 \). We consider two subcases.

**Subcase 2.1:** For each \( j \in N \setminus \{i, n\} \), \( x_j = 0 \). Then, \( y_{\{i,n\}} = \left( \frac{c_n}{2}, c_n - \frac{c_n}{2} \right) \).

Since \( 0 \leq x_i \leq c_i \) and \( x_n = c_n - x_i \geq c_n - c_i \geq 0 \), then \( x \in X(N, c) \). We consider two possibilities.
• \(i \neq n - 1\). Let \(\varepsilon \in \mathbb{R}\) be such that \(0 < \varepsilon < \min \left\{ c_{n-1} - \frac{c_i}{2}, \frac{c_i}{2}\right\}\). We show that agent \(i\) is better off announcing \(\frac{c_i}{2} - \varepsilon\) in Stage 1. Let \(x'\) be the proposal after agent \(i\)'s deviation. Then, \(x_i' = \frac{1}{2}c_i - \varepsilon\), \(x_n' = c_n - x_i'\), and for each \(j \in N \setminus \{i, n\}\), \(x_j' = 0\). Since \(y_n = c_n - \frac{c_i}{2}\),

\[
\tau_{n-1}^{n-1}(c, x') = \frac{1}{2} \left\{ c_{n-1} - \left(\frac{1}{2}c_i - \varepsilon\right) \right\}
\]

and

\[
\tau_n^{n-1}(c, x') = x_{n-1}' + x_n' - \tau_{n-1}^{n-1}(c, x')
\]

\[
= c_n - \frac{1}{2}c_i + \varepsilon - \tau_{n-1}^{n-1}(c, x')
\]

\[
= c_n - \frac{1}{2}c_i + \frac{1}{2} \left[ \varepsilon - \left( c_{n-1} - \frac{1}{2}c_i \right) \right]
\]

\[
= y_n + \frac{1}{2} \left[ \varepsilon - \left( c_{n-1} - \frac{1}{2}c_i \right) \right]
\]

\[
< y_n.
\]

Since taking agent \(n - 1\), agent \(n\) ends up with contribution \(\tau_n^{n-1}(c, x') < \tau_n^i(c, x) = y_n\), she does not take agent \(i\) in Stage 2. Since agent \(i\) ends up with contribution \(x_i' < x_i\), she is better off deviating, which violates the assumption that \(g\) is a SPE.

• \(i = n - 1\). Let \(j \in N \setminus \{n - 1, n\}\) and \(\varepsilon \in \mathbb{R}\) be such that \(0 < \varepsilon < \frac{1}{2} \min \{c_j, c_{n-1}\}\). We show that agent \(n - 1\) is better off announcing \(\frac{1}{2}c_{n-1} - \varepsilon\) in Stage 1. Let \(x'\) be the proposal after agent \((n - 1)\)'s deviation. Then, \(x_{n-1}' = \frac{1}{2}c_{n-1} - \varepsilon\), \(x_n' = c_n - x_{n-1}'\), and for each \(j \in N \setminus \{n - 1, n\}\), \(x_j' = 0\). Note that

\[
\tau_j^{i}(c, x') = \frac{1}{2} \min \left\{ c_j, \cdots, c_{n-2}, c_{n-1} - \left(\frac{1}{2}c_{n-1} - \varepsilon\right) \right\}
\]

\[
= \frac{1}{2} \min \left\{ c_j, c_{n-1} - \left(\frac{1}{2}c_{n-1} - \varepsilon\right) \right\}
\]

\[
= \min \left\{ \frac{1}{2}c_j, \frac{1}{4}c_{n-1} + \frac{1}{2}\varepsilon \right\}.
\]

If \(\tau_j^{i}(c, x') = \frac{1}{2}c_j\), then \(\tau_j^{i}(c, x') = c_n - \frac{1}{2}c_i + \varepsilon - \frac{1}{2}c_j < y_n\). If \(\tau_j^{i}(c, x') = \frac{1}{4}c_{n-1} + \frac{1}{2}\varepsilon\), then \(\tau_j^{i}(c, x') = c_n - \frac{1}{2}c_i + \frac{1}{2}\left(\varepsilon - \frac{1}{2}c_{n-1}\right) < y_n\). In either case,
Subcase 2.2: For some \( j \in N \setminus \{i, n\} \), \( x_j > 0 \). Let \( j \in N \setminus \{i, n\} \) be such that \( x_j > 0 \). Note that in stage 1, by subgame perfection, for each 
\( k \in N \setminus \{i, n\}, y_n \leq \tau_n^k(c, x) \). We consider two possibilities.

- For each \( k \in N \setminus \{i, n\}, y_n < \tau_n^k(c, x) \). Let \( \varepsilon \in \mathbb{R} \) be such that \( 0 < \varepsilon < \min \{\tau_n^i(c, x) - y_n, x_j\} \). We show that agent \( j \) is better off announcing \( x_j - \varepsilon \) in Stage 1. Let \( x' \) be the proposal after agent \( j \)'s deviation. Then, \( x'_n = x_n + \varepsilon \), and for each \( k \in N \setminus \{j, n\}, x'_k = x_k \). It can be shown that \( \tau_n^i(c, x) \leq \tau_n^i(c, x') \) and \( \tau_j^i(c, x) = \tau_j^i(c, x') \). Since \( x'_n = x_j + x_n \) and \( \tau_j^i(c, x) = \tau_j^i(c, x') \), then choosing agent \( i \), agent \( n \) ends up with contribution \( \tau_n^i(c, x') < \tau_n^i(c, x) \), which implies that agent \( n \) does not choose agent \( j \) in Stage 2. Since agent \( j \) ends up with contribution \( x_j - \varepsilon < x_j \), she is better off deviating, which violates the assumption that \( g \) is a SPE.

- For some \( k \in N \setminus \{i, n\}, y_n = \tau_n^k(c, x) \). Let \( k \in N \setminus \{i, n\} \) be such that \( y_n = \tau_n^k(c, x) \). If \( y_{i,n} = x_{i,n} \), then \( y = x \). By a similar argument as in Case 1, we conclude that \( y = Nu(N, c) \). Suppose that \( y_{i,n} \neq x_{i,n} \). Since \( y_{N \setminus \{i,n\}} = x_{N \setminus \{i,n\}} \) and \( \sum_{k \in N} y_k = \sum_{k \in N} x_k, y_i + y_n = x_i + x_n \). In Stage 2, by subgame perfection, \( y_n \leq x_n \). Since \( y_{i,n} \neq x_{i,n} \) and \( y_i + y_n = x_i + x_n \), \( y_n < x_n \) and \( x_i < y_i \). Let \( \varepsilon \in \mathbb{R} \) be such that \( 0 < \varepsilon < y_i - x_i \). We now show that agent \( i \) is better off announcing \( x_i + \varepsilon \). Let \( x'' \) be the proposal after agent \( i \)'s deviation. Then, \( x''_i = x_i + \varepsilon \), \( x''_n = x_n - \varepsilon \), and for each
\( j \in N \setminus \{i, n\}, x'_j = x_j \). It can be shown that \( \tau^k_k (c, x) - \frac{1}{2} \varepsilon \leq \tau^k_k (c, x''') \) and \( \tau^i_i (c, x) = \tau^i_i (c, x'''') \). Since \( x_i + x_n = x'''' + x_n'' \), \( \tau^i_n (c, x) = \tau^i_n (c, x''') \). Note that

\[
\tau^k_n (c, x''') = x'_k + x_n'' - \tau^k_k (c, x''') \leq x_k + x_n - \varepsilon - \tau^k_k (c, x) + \frac{1}{2} \varepsilon = x_k + x_n - \frac{\varepsilon}{2} - \tau^k_k (c, x) = \tau^i_n (c, x) - \frac{\varepsilon}{2} = y_n - \frac{\varepsilon}{2} < y_n = \tau^i_n (c, x) = \tau^i_n (c, x''') \]

which implies that choosing agent \( k \), agent \( n \) ends up with contribution \( \tau^k_n (c, x''') < \tau^i_n (c, x''') \). Thus, agent \( n \) does not choose agent \( i \) in Stage 2. Since agent \( i \) ends up with contribution \( x_i + \varepsilon < y_i \), she is better off deviating, which violates the assumption that \( g \) is a SPE.

By Subcases 2.1 and 2.2, \( y = Nu (N, c) \). Q.E.D.

Arin et al. (2009) provide a strategic justification for the nucleolus by proposing a \( n \)-stage sequential game in the spirit of Serrano (1995)'s game. In Stage 1, agent \( n \) announces a contributions vector \( x \). If \( x \) is unanimously accepted, it is the outcome; otherwise, bilateral negotiations take place. Responders reply sequentially according to a specified order on the set of responders. Acceptors contribute their components of \( x \) and each rejector \( j \) negotiates her contribution with agent \( n \) by invoking a particular rule to solve the reduced problem underlying the definitions of \textit{RS bilateral consistency} and \textit{RS converse consistency}. They show that there is a unique Nash equilibrium outcome of their game, which is the allocation chosen by the nucleolus.

Our game differs from theirs: (1) In Stage 1, each agent \( i \in N \setminus \{n\} \) announces a number \( x_i \) and agent \( n \) is associated with a number \( x_n \equiv c_n - \)
\[ \sum_{j \in N \setminus \{n\}} x_j; \] (2) Our results neither rely on any specific order on the set of responders nor any particular rule in solving two-agent reduced problems; and (3) Our equilibrium concept is SPE.

References


