Alternative Objectives of Oligopoly: An Aggregative Game Approach

Richard Cornes * Jun-ichi Itaya†

preliminary June 27, 2012

Abstract

The rapidly growing literature analyzes models in which firms maximize objects other than profit and enjoy market power. Examples include the labour-managed firm, mixed oligopoly, public/private partnerships and delegation models. These models typically retain the aggregative structure of the conventional Cournot model of imperfect competition. We exploit this fact and apply the framework recently developed by Cornes and Hartley (2005, 2007, 2012) to analyze the properties of equilibrium in such games. We show that existing treatments often make more restrictive assumptions than necessary in order to generate their results. Specifically, we identify conditions sufficient to ensure existence of a unique equilibrium, and we explore their comparative static properties.

Keywords: Aggregative Game, Oligopoly, Hahn’s Condition, Non-profit Maximization, Share Function

JEL classifications: C72, D43, L21

---

*F. H. Gruen Professor of Economics, Research School of Economics, Australian National University. E-mail address: rccornes@aol.com.

†Corresponding author: Graduate School of Economics and Business Administration, Hokkaido University, Sapporo, 060-0809, JAPAN. Tel: +81-11-706-2858; fax: +81-11-706-4947. E-mail address: itaya@econ.hokudai.ac.jp.
1 Introduction

Traditional economic theories of firms assume that the sole aim of firms is profit maximization. However, this assumption has been criticized by management researchers and economists who have studied the organization and modern corporations on the ground that the complexity of managerial decision process and the separation of ownership and control, which is common to most large scale corporations, would lead to managers’ nonprofit-maximizing behavior. If firms are likely to move away from profit maximizing behavior, what is the alternative? In the 80’s Vickers (1985), Fershtman (1985), Fershtman and Judd (1987) and Sklivas (1987) (hereafter, VFJS for short) examine a two-stage delegation game in which owners in the first stage maximize profit by properly choosing the weights on profit and sale revenue in the manager’s incentive contract, and managers in the second stage maximize the objective function with the weights chosen in the first stage. They have shown that a manager’s decision based partly on non-profit considerations like sales can be more profitable for owners than the decisions driven solely by profit. The papers of VFJS led to fruitful consecutive studies in which researchers adopted the basic structure of the delegation game in their examinations. Fumas (1992) and Miller and Pazgal (2002) among them consider a two-stage delegation game in which the manager’s incentive contract is based on a weighted sum of the firm’s own profit and its rival profits (or their average profits). More recently, Jansen, et al. (2007) and Ritz (2008) further consider the delegation game whose managerial incentive scheme is a weighted average between profits and market share. This literature usually assumes not only that the manager’s incentive contract is composed of a linear combination of profit and a non-profit indicator but also oligopolistic competition with linear demand and linear/quadratic production cost functions.

While there is a growing and large literature on mixed oligopoly that examines the interaction between private and public firms, this literature generally assumes that the public firm maximizes social welfare, defined as the sum of consumer surplus and firm profit. This literature has been used to examine the welfare consequences of strategic trade policy, privatization, open door policy and so on (see, e.g., DeFraja and Delbono, 1989; Fjell and Pal,
1996; White, 1996). The mixed oligopoly models have been extended to the models having a variety of the other objective functions of firms. Barros (1995) and White (2001) have applied the VFJS’s analysis to mixed oligopoly models and shown how the presence of managerial incentives make the resultant outcomes differ in traditional public/private mixed oligopolies. White (2002) considers a two-stage game in a mixed oligopoly setting where a government maximizes its true objective function by assigning the weights on a generalized social welfare function and a public firm in the second stage maximizes the objective function with the weights chosen in the first stage.

Since most of the papers in these two literatures also have assumed linear inverse demand and linear or quadratic production cost functions, each oligopoly firm has a linear reaction function which makes the issues such as the existence, uniqueness and stability of a Cournot-Nash equilibrium in oligopolistic competition being secondary importance. Such simplification severally limits the validity of the results of their models. In order to assess the robustness of their comparative statics, we need to investigate an oligopoly model under more general demand and cost functions. However, allowing for more general demand and cost functions, the resulting oligopoly model may exhibit multiple equilibria, which in turn significantly reduces the predictability of comparative statics properties. There are two exceptional papers which are concerned with these issues. Based on the model of Kaneda and Matusi (2003) in which each firm’s objective is to maximize the weighted average of profit and other objectives such as revenue, market share, Kato and Tomaru (2007) investigate the effects of subsidy in mixed oligopoly. They have assumed the conditions which are stronger Hahn’s condition coupled with the additional conditions on the other objective function to ensure the unique existence of Nash equilibrium. Roy (2009) proves the existence and uniqueness of the equilibrium in the mixed oligopoly model in which a public firm maximizes a weighted average of share-holders utilities and consumers’ surplus.

The principle motivation of this paper is to fill this gap in the existing literatures on mixed oligopoly and oligopoly models with delegation. To this end, we set up a Cournot oligopoly model composed of $n$ heterogenous firms each of which maximizes the different weighted average of profit and another objective such as revenue, market share, per-worker profits,
social welfare and so on.¹ We adopt the weighted average form of the objective function suggested by Kaneda and Matusi (2003) not only because of its flexibility but also because their objective function shares a common structure with those in the models of the mixed oligopoly and the oligopoly models with a delegation game. In other words, their weighted sum formulation of the objective function allows us to capture various types of the objective functions for oligopoly firms ranging from a purely profit maximizing oligopoly firm to a purely non-profit firm by simply varying values of the weight. Nevertheless, their analysis is limited only on the case of homogenous oligopoly firms.

Another important novel of this paper is that we adopt the share function approach suggested by Cornes and Hartley (2005, 2007, 2012) to investigate the existence, uniqueness, stability and principle comparative statics properties of Cournot-Nash equilibrium in the above-mentioned oligopoly models. The share function approach of Cornes and Hartley is not only simple but also a powerful tool to handle these issues in a systematic and unified way. Irrespective of the form of the firm’s objective function this approach provides a general condition. Another advantage of this approach is to easily allow for many heterogenous oligopoly firms, which would be a generalization of Kanada and Matsui’s model. We show that our conditions are simpler, substantially weaker, and naturally satisfied by those oligopoly models that do not satisfy Hahn’s conditions. Although the existing literature on profit-maximizing oligopoly models has usually assumed the so-called Hahn’s condition in order to prove the existence and uniqueness of the Cournot-Nash equilibrium, this condition is neither sufficient nor necessary. Much weaker conditions can be stated which ensure the existence, uniqueness and global stability of Cournot-Nash equilibrium in oligopolistic competition.

The reminder of the paper is organized as follows. In Section 2 we present the basic model of oligopoly in which each firm maximizes the weighted average of profit and another object such as revenue, market share, per-worker profits and so on. We identify the general condition for the existence and uniqueness of the Cournot-Nash equilibrium under the oligopoly models having alternative objectives. In Section 3 we undertake a stability analysis of the oligopoly

¹To our best knowledge, the unpublished paper of Kaneda and Matsui (2003) is the first one that have introduced such a formulation. The resulting fairly complicated and restrictive technical conditions would not only limit the applicability of the model, but also obscure the main messages of their model.
model with alternative objectives. In Section 4 we investigate the conditions for existence, uniqueness and global stability of a Cournot-Nash equilibrium in the respective oligopoly models with specific alternative objectives. In Section 5 we conduct a comparative statics analysis. In Section 6 we apply the share function approach to the oligopoly models having multiple objectives more than two and non-linear managerial contracts. In Section 7 the concluding remarks are presented.

2 The Analysis

2.1 The Model

We assume that there are \( n \) potential producers of a homogeneous output, indexed by \( i = 1, 2, \ldots, n \) in Cournot oligopoly, but the set of active firms (i.e., firms producing a positive output) will be determined endogenously.\(^2\) Firms choose simultaneously their own output. The inverse demand function is given by \( P = P(X) \), where \( P \) is the price, \( x_i \) is the output of firm \( i \) and \( X = \sum_{j=1}^{n} x_j \) is industry output.

Define the profit function of firm \( i \) as follows:

\[
\pi_i(x_i, X) = P(X)x_i - C_i(x_i). \tag{1}
\]

We further define the true objective function of firm \( i \), which will be a main focus of our analysis, as

\[
\Pi_i(x_i, X, \theta_i) \equiv (1 - \theta_i) \pi(x_i, X) + \theta_i F_i(x_i, X),
\]

\[
= (1 - \theta_i) [P(X)x_i - C_i(x_i)] + \theta_i F_i(x_i, X), \tag{2}
\]

where \( \theta_i \in [0, 1] \) is the weight assigned on the objective other than profit. Note that when \( \theta_i = 0 \), firm \( i \) is a profit maximizer. Without loss of generality, we assume that that \( \theta_1 \leq 2\)\(^2\) This assumption does not imply that the number of firms is determined through the entry of new firms until profits are driven to zero, but that each firm will be an active supplier depending on the price of output and its cost function.
\[ \theta_2 \leq \cdots \leq \theta_n. \] The function \( F_i(\cdot) \) represents the other non-profit maximizing objective such as sales, social welfare, and so forth. In this paper we focus on the following objectives.

1. Revenue: \( F_i^R(x_i, X) \equiv P(X)x_i, \)
2. Output: \( F_i^O(x_i, X) \equiv x_i, \)
3. Market share: \( F_i^M(x_i, X) \equiv x_i/X, \)
4. Profit per worker: \( F_i^{LM}(x_i, X) \equiv \frac{P(X)x_i - C_i(x_i)}{N(x_i)}, \)
5. Relative profit: \( F_i^{RP}(x_i, X) \equiv \pi_i(x_i, X) - \frac{1}{n} \sum_{j=1}^{n} \pi_j(x_i, X), \)
6. Social welfare: \( F_i^{SW}(x_i, X) \equiv \int_0^X P(q) dq - P(X)X + \sum_{j=1}^{n} \pi_j(x_j, X). \)

Assume that

\[ A.1 \quad \Pi_i(x_i, X, \cdot) \in C^2 \text{ for } \forall i \in I, \] where \( I \) is the set of firms (assumed to be finite).

The first-order condition associated with the maximization of (2) is given by

\[ \frac{\partial \Pi_i(x_i, X, \theta_i)}{\partial x_i} = (1 - \theta_i) [P'(X)x_i + P(X) - C_i'(x_i)] + \theta_i [F_{i,1} + F_{i,2}] \leq 0, \quad (3) \]

with equality if \( x_i > 0 \), where \( \theta_i \) is a given parameter, and \( F_{i,1} \equiv \partial F_i(x_i, X) / \partial x_i \) and \( F_{i,2} \equiv \partial F_i(x_i, X) / \partial X \). We denote the middle expression in (3) as \( \gamma_i(x_i, X, \theta_i) \), which represents marginal profit.

### 2.2 Existence and Uniqueness

Cornes and Hartley (2005a, 2010) made the following assumptions in order to ensure existence and uniqueness of Nash equilibria in pure strategies:

\[ A.2 \quad \text{For for } \forall i \in I, \text{ if } (x_i, X) \text{ satisfies } x_i < \omega_i, 0 < x_i \leq X \text{ and } \gamma_i(x_i, X, \theta_i) = 0, \text{ then } \]

\[ \frac{\partial \gamma_i(x_i, X, \theta_i)}{\partial x_i} < 0, \]

where

\[ \frac{\partial \gamma_i(x_i, X, \theta_i)}{\partial x_i} \equiv (1 - \theta_i) [P'(X) - C''_i(x_i)] + \theta_i [F_{i,11} + F_{i,21}] \cdot (4) \]

and \( F_{i,11} \equiv \partial^2 F_i(x_i, X) / \partial x_i^2 \) and \( F_{i,21} \equiv \partial^2 F_i(x_i, X) / \partial X \partial x_i. \)
A.3 For ∀i ∈ I, if \((x_i, X)\) satisfies \(x_i < \omega_i, 0 < x_i < X\) and \(\gamma_i(x_i, X, \theta_i) = 0\), then

\[
x_i \frac{\partial \gamma_i(x_i, X, \theta_i)}{\partial x_i} + X \frac{\partial \gamma_i(x_i, X, \theta_i)}{\partial X} < 0,
\]

where

\[
\frac{\partial \gamma_i(x_i, X, \theta_i)}{\partial X} \equiv (1 - \theta_i) \left[ P''(X) x_i + P'(X) \right] + \theta_i \left[ F_{i,12} + F_{i,22} \right]
\]

and \(F_{i,12} \equiv \partial^2 F_i(x_i, X) / \partial x_i \partial X\) and \(F_{i,22} \equiv \partial^2 F_i(x_i, X) / \partial X_i^2\).

A.4 There exists at least one firm, say firm \(i\), such that for some \(x_i \in (0, \omega_i)\)

\[
\arg\max_{x_i \in [0, \omega_i]} \Pi_i(x_i, x_i, \theta_i) > 0.
\]

To understand the implications of A.2 and A.3, we need to know the concepts of the replacement and share functions. When firm \(i\) produces a positive amount of output, that is, an interior solution exists, since \(\gamma_i(.)\) is continuously differentiable and due to Assumption 3, we can apply the implicit function theorem to the first-order condition \(\gamma_i(\hat{x}_i, \hat{X}, \theta_i) = 0\). As a result, there exists a continuously differentiable function \(r_i: B \mapsto A\) with \(\gamma_i(x_i(X), X, \theta_i) = 0\), where \(A\) represents some open set containing \(\hat{X}\) and \(B\) represents some open set containing \(\hat{x}_i\). We call such a function a replacement function following Cornes and Hartley (2005a, 2007, 2010). More formally, we can define the replacement function as follows:

\[
b_{0}(X_i, \theta_i) = -\frac{\partial \gamma_i/\partial X}{(\partial \gamma_i/\partial x_i) + (\partial \gamma_i/\partial X)}.
\]

Since \((\partial \gamma_i/\partial x_i) + (\partial \gamma_i/\partial X) < 0\) due to the requirement of the second-order condition,

\[
b_{0}(X_i, \theta_i) = -\frac{\partial \gamma_i/\partial X}{(\partial \gamma_i/\partial x_i) + (\partial \gamma_i/\partial X)} < 0 \text{ if and only if } \frac{\partial \gamma_i}{\partial X} < 0.
\]

In words, the slope of the best-response function depends on the sign of \((\partial \gamma_i/\partial X)\).
Definition 1  If for any value of $X \geq 0$, there is a unique value $x_{i}^{BR} = r_{i}(X, \theta_{i})$ such that $x_{i}^{BR}$ is a best response to $X - x_{i}^{BR}$, then the function $r_{i}(X, \theta_{i})$ is the replacement function of firm $i$.

For the reason stated later, when proving uniqueness and existence of a Cournot-Nash equilibrium, it may be more convenient to work with the share function rather than the replacement function. Firm $i$’s share function is defined as follows:

Definition 2  Let firm $i$ have a replacement function $x_{i} = r_{i}(X, \theta_{i})$. Then, for all $X > 0$, the function $s_{i}(X, \theta_{i}) = r_{i}(X, \theta_{i})/X$ is the share function of firm $i$.

A Cournot-Nash equilibrium of the oligopoly model is an allocation at which $\sum_{j=1}^{n} s_{j}(X, \theta_{i}) = 1$. More precisely, if (i) each firm’s share function is continuous and monotonic decreasing so long as the share value is strictly positive, (ii) all share functions approach or equal zero for larger $X$, and (iii) each firm’s share function has an least upper bound of unit, then it follows that there exists a unique Cournot-Nash equilibrium in oligopolistic competition. The condition for monotonic decreasing is given by

$$ \frac{ds_{i}(X, \theta_{i})}{dX} = - \frac{x_{i} \frac{\partial r_{i}(x_{i}, X, \theta_{i})}{\partial x_{i}} + X \frac{\partial r_{i}(x_{i}, X, \theta_{i})}{\partial X}}{X \frac{\partial r_{i}(x_{i}, X, \theta_{i})}{\partial x_{i}}} < 0 \text{ if and only if } A2 \text{ and } A3 \text{ hold.} $$

Since the share function of firm $i$ is bounded above by $\omega_{i}/X$, all share functions approach or equal to zero for large enough $X$, and therefore the same is true of the aggregate share function. Hence, (ii) is satisfied under $A2$ and $A3$. We conclude that a necessary and sufficient condition of an active equilibrium where at least one firm produces a positive amount of output is that the aggregate share function should be equal to or exceed one for some value of $X > 0$, which is implied by $A.4$.

There are several advantages in using the share function. The first advantage is that it allows us to work with functions defined on the real line rather than a multi-dimensional best-reply mapping from $\mathbb{R}^{n}$ to $\mathbb{R}^{n}$. The second one is that even if the replacement function is not monotonic, the share function may be monotonically decreasing; in this sense, the
share function approach would potentially provide weaker conditions for the existence and uniqueness for a Cournot-Nash equilibrium (see footnote 3).

In order to ensure the existence and uniqueness of Cournot-Nash equilibrium several authors of the existing literature usually assume Hahn’s condition associated with dynamic stability (Hahn, 1962). Hahn’s condition requires that as $X_{-i}$ increases, the replacement function $x_i(X, \theta_i)$ decreases. Cornes and Hartley’s condition allows for the possibility that $x_i(X, \theta_i)$ increases in response to an increase in $X_{-i}$. In other words, their condition does not request submodularity or strategic substitutes. For this reason, Cornes and Hartley’s condition is significantly weaker than Hahn’s condition.\(^5\)

We can prove the following theorem:

**Proposition 1** Assume that A.1—A.4 hold. There exists a unique Cournot-Nash equilibrium in the Cournot oligopoly game in which each firm maximizes (2).

**Proof.** The proof is found in Cornes and Hartley (2005b, 2010). \(\blacksquare\)

It should be important to emphasize that even if Hahn’s condition is not satisfied, there may be a Nash equilibrium.

**Remark 1** Kaneda and Matsui (2003) have assumed Hahn’s conditions to ensure the existence and uniqueness of Nash equilibrium of their oligopoly model with identical cost functions, that is,

$$P''(X)x_i + P'(X) \leq 0 \text{ and } C''(x_i) - P'(X) > 0.$$  

Furthermore, in order to satisfy the second order condition for the profit function they assume

$$P''(X)x_i + 2P'(X) - C''(x_i) < 0.$$  

These conditions request that the best-reply functions of all firms are downward sloping. They further assume that

$$F_1 \leq 0, F_{12} + F_{22} < 0 \text{ and } F_{11} + 2F_{12} + F_{22} < 0.$$  

\(^5\)For example, consider the constant elasticity demand function $X = aP^{-\varepsilon}$ which does not satisfy Hahn’s condition (i.e., $P''(X)x_i + P'(X) < 0$).
Although they need all of these conditions together to ensure the existence of Nash equilibrium, our conditions for the existence and uniqueness is not only much simpler, but also allow for heterogeneity of firms (i.e., heterogenous cost functions).

**Remark 2** We need not to assume that the second-order condition is satisfied. Indeed, our assumption A.3 gives the second-order condition. While the second-order condition is given by

\[
(1 - \theta_i) [P''(X) x_i + P'(X) + P'(X) - C''_i(x_i)] \\
+ \theta_i [F_{i,11}(x_i, X) + 2F_{i,12}(x_i, X) + F_{i,22}(x_i, X)] < 0, \tag{6}
\]

A.3 implies that

\[
\frac{\partial \gamma_i(x_i, X)}{\partial x_i} + \frac{\partial \gamma_i(x_i, X)}{\partial X} = \\
(1 - \theta_i) [P'(X) - C''_i(x_i)] + \theta_i [F_{i,11}(x_i, X) + F_{i,21}(x_i, X)] + \\
(1 - \theta_i) P''(X) x_i + \theta_i [F_{i,12}(x_i, X) + F_{i,22}(x_i, X)]
\]

\[
= (1 - \theta_i) [P'(X) - C''_i(x_i) + P''(X) x_i] + \\
\theta_i [F_{i,11}(x_i, X) + 2F_{i,21}(x_i, X) + F_{i,22}(x_i, X)] < 0,
\]

which coincides with the second-order condition (6).

### 3 Comparative Statics

In this section we investigate comparative statics properties under a variety of alternative objectives of oligopoly firms, which have appeared in the literatures on delegation and mixed oligopoly, as stated in the introduction. Since some of the objective functions can be viewed as the second stage of the delegation game analyzed by VFJS, we need a further analysis to ensure the uniqueness of every step of their delegation game. To do this, we need the following two propositions demonstrated by Cornes and Hartely (2012), which is a main tool to derive
comparative statics results in the subsequent sections.

**Proposition 2** [Lemma 6.1 of Cornes and Hartely (2012)] Consider players $i, j \in I$ in an aggregative game in which all firms are regular with (unique) active equilibrium $\hat{x} = (x_1, x_2, ..., x_n)$. Suppose that, for $k = i, j$, we have $\gamma_k(x_k, X) = \phi_k(x_k, X)\hat{\gamma}_k(x_k, X)$, where $\phi_k(x_k, X) > 0$ and $\hat{\gamma}_i(x_i, X) < \hat{\gamma}_j(x_j, X)$ for any $(x, X)$ satisfying $0 < x \leq X$ and $x < \min\{\omega_i, \omega_j\}$.

1. If $\hat{x}_j = 0$, then $\hat{x}_i = 0$.
2. If $\hat{x}_j \in (0, \omega_j)$, then $\hat{x}_i < \hat{x}_j$.

This lemma implies that, if one player’s marginal profit exceeds another’s, the former choose a larger equilibrium output, if it is feasible.

**Proposition 3** [Theorem 8.1 of Cornes and Hartely (2012)] Let $G^1$ and $G^2$ be aggregative games with the same (finite) set of active firms, $I$, the same strategy sets and the same profits for all firms except $i \in I$. Suppose that $G^1$ and $G^2$ both have (unique) Nash equilibria $\hat{x}^1$ and $\hat{x}^2$. Suppose further that, for $k = 1, 2$, and any $(x_i, X)$ satisfying $0 < x_i \leq X$ and $x_i < \omega_i$, we have $\gamma^k_i(x_i, X) = \phi^k_i(x_i, X)\hat{\gamma}^k_i(x_i, X)$, where $\phi^k_i(x_i, X) > 0$ and $x_i < \omega_i$, we have $\hat{\gamma}^k_i(x_i, X) < \hat{\gamma}^k_j(x_j, X)$. If $\hat{x}^2_i > 0$, $\hat{x}^1_i < \omega_i$, then

1. $\hat{X}^2 > \hat{X}^1$,
2. firms in $I \setminus \{i\}$ that are inactive in $G^1$ are inactive in $G^2$,
3. if $j \neq i$ and $\Pi_j(x_j, X)$ is strictly decreasing (resp., increasing) in $X$ for all $x_j \in (0, \omega_j]$, active firms in $G^1$ are worse (resp., better) off in $G^2$ than in $G^1$, and
4. if $\Pi^1_i(0, X) \leq \Pi^2_i(0, X)$ for all $X > 0$, then firm $i$ is better off in $G^2$ than in $G^1$.

**3.1 Revenue**

VFJS consider the oligopoly model with delegation in which the manager’s incentive contract consists of a linear combination of profits and sale revenues. In this case, each oligopoly firm
is to maximize the delegated objective function such as:

$$\Pi^R_i(x_i, X, \theta_i) \equiv (1 - \theta_i) [P(X)x_i - C_i(x_i)] + \theta_i P(X)x_i. \quad (7)$$

The first order condition for profit maximization is given by

$$\gamma^R_i(x_i, X, \theta_i) \equiv (1 - \theta_i) [P(X) + P'(X)x_i - C''_i(x_i)] + \theta_i P(X) + P'(X)x_i \leq 0, \quad (8)$$

where $\gamma^R_i(x_i, X, \theta_i)$ stands for the marginal profit function as a function of $x_i$ and $X$. To ensure existence and uniqueness of a Cournot-Nash equilibrium, we assume

$$\frac{\partial \gamma^R_i(x_i, X, \theta_i)}{\partial x_i} = (1 - \theta_i) [P'(X) - C''_i(x_i)] + \theta_i P'(X) < 0, \quad (9)$$

and

$$x_i \frac{\partial \gamma^R_i(x_i, X, \theta_i)}{\partial x_i} + X \frac{\partial \gamma^R_i(x_i, X, \theta_i)}{\partial X} = x_i [(1 - \theta_i) [P'(X) - C''_i(x_i)] + \theta_i P'(X)] + X [P''(X)x_i + P'(X)] < 0. \quad (10)$$

To perform meaningful comparative statics exercises, in what follows we further assume

**A.5** There exists a $\xi \in (0, \infty)$ such that $P(X) > 0$ for $X \in [0, \xi)$, and $P(X) = 0$ holds for $X \in [\xi, \infty)$. Moreover, $P'(X) \leq 0$ for $X \in [0, \xi)$.

**A.6** For $\forall i \in I$, $C''_i(x_i) \geq 0$ for any $x_i \geq 0$ with strictly inequality if $x_i > 0$, and $C_i(0) = 0$.

Applying Propositions 2 and 3, we obtain

**Proposition 4** Assuming that **A.1—A.6**, we have the following.

1. Any positive shock of one of more of the firms (i.e., an increase in $\theta_i$) leads to an increase
in the equilibrium amount of industry output, the profits of the other firms are reduced, and the profit of firm $i$.

3. $\hat{x}_1 \leq \hat{x}_2 \leq \cdots \leq \hat{x}_n$, and

4. if $\theta_i < \theta_j$ for any $i < j$ and $\hat{x}_j > 0$, then $\hat{x}_i < \hat{x}_j$.

**Proof.** To prove Statements 1 and 2, we differentiate (8) with respect to $\theta_i$ to yield

$$
\frac{\partial \gamma_i^R(x_i, X, \theta_i)}{\partial \theta_i} = -[P(X) + P'(X)x_i - C'_i(x_i)] + P(X) + P'(X)x_i = C'_i(x_i) > 0,
$$

which implies that $\gamma_i^R(x_i, X, \theta_i) < \gamma_i^R(x_i, X, \tilde{\theta}_i)$ for $\theta_i < \tilde{\theta}_i$. Proposition 3 implies that if firm $i$ is active in the equilibrium of $G^2$, and thus $\hat{X}$ increases. That is, when a firm puts a smaller weight on profit, it leads to an increase in industry output (and thus a fall in price).

Furthermore, since $P'(X) < 0$,

$$
\frac{\partial \Pi_j^R(x_j, X, \theta_j)}{\partial X} = (1 - \theta_j)P'(X)x_j + \theta_jP'(X)x_j = P'(X)x_j < 0, \forall j \in I \text{ except for } i,
$$

which implies that $\Pi_j^R(x_j, X, \theta_j)$ is strictly decreasing in $X$ for all $x_j \in (0, \omega_j]$. Hence, it follows from statement 2 of Proposition 3 that the active firms other than $i$ are made worse off by this change, that is, the profits of the firms other than $i$ decrease in response to this change. Next, since

$$
\Pi_i^R(0, X, \theta_i) = 0 = \Pi_i^R(0, X, \tilde{\theta}_i)
$$

for all $X$, where noting that $\tilde{\theta}_i > \theta_i$, it follows from statement 4 of Proposition 3 that this change increases the profit of firm $i$. Moreover, since $\theta_j \geq \theta_i$,

$$
\gamma_i^R(x, X, \theta_i) - \gamma_j^R(x, X, \theta_j) = \{(1 - \theta_i) - (1 - \theta_j)\}[P(X) + P'(X)x - C'(x)]
$$

$$
+ (\theta_i - \theta_j)[P(X) + P'(X)x],
$$

$$
= (\theta_j - \theta_i) [P(X) + P'(X)x - C'(x) - P(X) - P'(X)x]
$$

$$
= (\theta_j - \theta_i) [-C'(x)] \leq 0.
$$
Hence, it follows from Proposition 2 that \( \hat{x}_i \leq \hat{x}_j \). The strict inequality follows when \( \theta_i < \theta_j \).

3.2 Output

The second model is that the manager’s incentive contract consists of a linear combination of profits and units sold (i.e., output) as considered by Vickers (1985) and Stewert (1992). In this case, the manager’s objective function of oligopoly firm \( i \) is expressed by:

\[
\Pi_i^O (x_i, X, \theta_i) = (1 - \theta_i) [P(X) x_i - C_i(x_i)] + \theta_i x_i. \tag{12}
\]

The first order condition for profit maximization is given by

\[
\gamma_i^O (x_i, X, \theta_i) = (1 - \theta_i) [P(X) + P'(X) x_i - C'_i(x_i)] + \theta_i \leq 0. \tag{13}
\]

We assume

\[
\frac{\partial \gamma_i^O (x_i, X, \theta_i)}{\partial x_i} = (1 - \theta_i) [P'(X) - C''_i(x_i)] < 0, \tag{14}
\]

and

\[
x_i \frac{\partial \gamma_i^O (x_i, X, \theta_i)}{\partial x_i} + X \frac{\partial \gamma_i^O (x_i, X, \theta_i)}{\partial X} = x_i (1 - \theta_i) [P'(X) - C''_i(x_i)] + X (1 - \theta_i) [P'''(X) x_i + P'(X)] < 0. \tag{15}
\]

To see how an increase affects the behavior of oligopoly firms, we differentiate (13) with respect to \( \theta_i \) to yield

\[
\frac{\partial \gamma_i^O (x_i, X, \theta_i)}{\partial \theta_i} = - [P(X) + P'(X) x_i - C'_i(x_i)] + 1 > 0, \tag{16}
\]

whose positive sign follows from (13). This implies that \( \hat{X} \) increases. Furthermore, since

\[
\frac{\partial \Pi_j^O (x_j, X, \theta_j)}{\partial X} = (1 - \theta_i) P'(X) x_i < 0, \forall j \in I \text{ except for } i,
\]

13
\( \Pi_j^O (x_j, X, \theta_j) \) is strictly decreasing in \( X \) for all \( x_j \in (0, \omega_j] \), and thus the profits of the active firms other than \( i \) fall in \( \theta_i \). On the other hand, since

\[
\Pi_i^O (0, X, \theta_i) = 0 = \Pi_i^O \left( 0, X, \tilde{\theta}_i \right) \text{ for all } X,
\]

it follows from Proposition 2 that this change increases the profit of firm \( i \). Finally, since \( \theta_i \leq \theta_j \),

\[
\gamma_i^O (x, X, \theta_i) - \gamma_j^O (x, X, \theta_j) = (\theta_j - \theta_i) [P(X) + P'(X)x - C'(x_i) - 1] \leq 0,
\]

which implies that \( x_i \leq x_j \), while the strict inequality follows when \( \theta_i < \theta_j \).

### 3.3 Market Share

Jansen et al. (2007) and Ritz (2008) consider a delegation game in which the manager’s incentive contract is written as a linear combination of profits and the market share. In this case, the manager’s objective function can be expressed by:

\[
\Pi_i^M (x_i, X, \theta_i) \equiv (1 - \theta_i) [P(X)x_i - C_i(x_i)] + \theta_i (x_i/X). \tag{17}
\]

The first order condition for profit maximization is given by

\[
\gamma_i^M (x_i, X, \theta_i) \equiv (1 - \theta_i) [P(X) + P'(X)x_i - C_i'(x_i)] + \theta_i (X - x_i)X^{-2} \leq 0. \tag{18}
\]

To ensure existence and uniqueness of a Cournot-Nash equilibrium, we assume

\[
\frac{\partial \gamma_i^M (x_i, X, \theta_i)}{\partial x_i} = (1 - \theta_i) [P'(X) - C_i''(x_i)] - \theta_i \frac{1}{X^2} < 0, \tag{19}
\]
\[ x_i \frac{\partial \gamma_i^M (x_i, X, \theta_i)}{\partial x_i} + X \frac{\partial \gamma_i^M (x_i, X, \theta_i)}{\partial X} = x_i \left\{ (1 - \theta_i) [P' (X) - C''_i (x_i)] - \theta_i \frac{1}{X^2} \right\} + X \left\{ (1 - \theta_i) [P'' (X) x_i + P' (X)] + \theta_i \frac{1}{X^3} (-X + 2x_i) \right\}, \]

\[ = x_i (1 - \theta_i) [P' (X) - C''_i (x_i)] + X (1 - \theta_i) [P'' (X) x_i + P' (X)] + \theta_i \frac{x_i - X}{X^2} < 0. \tag{20} \]

To get meaningful comparative statics results, we differentiate (18) with respect to \(\theta_i\) to yield

\[ \frac{\partial \gamma_i^M (x_i, X, \theta_i)}{\partial \theta_i} = -[P (X) + P' (X) x_i - C'_i (x_i)] + \frac{X - x_i}{X^2} > 0, \tag{21} \]

whose positive sign immediately follows from (18). This implies that \(X\) increases in response to this change. Furthermore, since

\[ \frac{\partial \Pi_j^M (x_j, X, \theta_i)}{\partial X} = (1 - \theta_j) P' (X) x_j - \theta_j (x_j / X^2) < 0, \forall j \in I \text{ except for } i, \]

\(\Pi_j^M (x_j, X, \theta_i)\) is strictly decreasing in \(X\) for all \(x_j \in (0, \omega_j]\); hence, the profits of the firms other than \(i\) fall in response to this change. Since

\[ \Pi_i^M (0, X, \theta_i) = 0 = \Pi_i^M \left( 0, X, \tilde{\theta}_i \right) \text{ for all } X, \]

the profit of firm \(i\) increases in \(\theta_i\). Lastly, we have

\[ \gamma_i^M (x, X, \theta_i) - \gamma_j^M (x, X, \theta_j) = [(1 - \theta_i) - (1 - \theta_j)] [P (X) + P' (X) x - C' (x)] \\
+ (\theta_i - \theta_j) (X - x) X^{-2} \]
\[
= (\theta_j - \theta_i) \left[ P(X) + P'(X)x - C'(x) - \frac{X-x}{X^2} \right] \leq 0,
\]
whose negative sign follows from (18). This implies that \( x_i \leq x_j \) for \( \theta_i \leq \theta_j \), while the strict inequality follows when \( \theta_i < \theta_j \).

### 3.4 Relative Profit

Fumas (1992) and Miller and Pazgal (2002) consider the delegation game in which the manager’s incentive contract is written as the weighted sum of the firm’s own profit and the rival firm’s profit (i.e., the difference between own profit and the profits of the firm’s rival).\(^7\) Given this linear contract, the manager’s objective function of oligopoly firm \( i \) can be expressed by

\[
\Pi_{i}^{RP}(x_1, \ldots, x_n, X, \theta_i) \equiv (1 - \theta_i) \pi_i(x_i, X) + \theta_i \left[ \pi_i(x_i, X) - \frac{1}{n} \sum_{j=1}^{n} \pi_j(x_i, X) \right],
\]

\[
= \pi_i(x_i, X) + \theta_i \left[ -\frac{1}{n} \sum_{j=1}^{n} \pi_j(x_i, X) \right],
\]

(22)

where \( \pi_i(x_i, X) - (1/n) \sum_{j=1}^{n} \pi_j(x_i, X) \) represents the difference between firm \( i \)'s own profit and the average industry profit. The first order condition for profit maximization is given by

\[
\gamma_i^{RP}(x_i, X, \theta_i) \equiv P'(X)x_i + P(X) - C'_i(x_i) + (\theta_i/n) [-P'(X)X + C'_i(x_i)] \leq 0. \quad (23)
\]

To ensure existence and uniqueness of a Cournot-Nash equilibrium, we assume

\[
\frac{\partial \gamma_i^{RP}(x_i, X, \theta_i)}{\partial x_i} = P'(X) - \left( 1 - \frac{\theta_i}{n} \right) C''_i(x_i) < 0,
\]

\(^7\)Schaffer (1980) and Vega-Redondo (1987) also consider the relative profit maximization objective of the firm which is defined to be the difference between the firm’s own (absolute) profit and the average profit of all the firms, although their objective function is not the weighted sum of these two types of profits. Choi (2006), on the other hand, considers a mixed oligopoly model where private firms are maximizers of either (absolute) profits or relative profits, while the public firm maximizes the sum of social welfare and either (absolute) profits or relative profits.
and
\[
\frac{\partial \gamma_{RP}^{i}(x_i, X, \theta_i)}{\partial x_i} + X \frac{\partial \gamma_{RP}^{RP}(x_i, X, \theta_i)}{\partial X} = x_i \left[ P'(X) - \left(1 - \frac{\theta_i}{n}\right) C''_i(x_i)\right] + X \left[ P''(X) (x_i - X) + P'(X) \left(1 - \frac{\theta_i}{n}\right)\right] < 0. \tag{24}
\]

Moreover, differentiating the RHS of (23) with respect to \(\theta_i\) yields
\[
\frac{\partial \gamma_{RP}^{i}(x_i, X, \theta_i)}{\partial \theta_i} = \frac{1}{n} \left[P'(X) X - C'_i(x_i)\right] < 0, \tag{25}
\]
which implies that \(X\) increases. Furthermore,
\[
\frac{\partial \Pi_{RP}^{j}(x_j, X, \theta_j)}{\partial X} = P'(X) \left[x_j - \frac{\theta_j}{n} \sum_{k=1}^{n} x_k\right] = P'(X) \left[x_j - \frac{\theta_j X}{n}\right] \geq 0, \forall j \in I \text{ except for } i,
\]
which implies that \(\Pi_{RP}^{j}(x_j, X, \theta_j)\) may be decreasing or decreasing in \(X\) for all \(x_j \in (0, \omega_j]\).

Hence, whether the profits of the firms other than \(i\) may increase or decrease depends on whether firm \(i\)'s own output is greater or smaller than the average industry output weighted by \(\theta_j\). Since
\[
\Pi_{RP}^{i}(0, X, \theta_i) = 0 = \Pi_{RP}^{i}(0, X, \tilde{\theta}_i) \text{ for all } X,
\]
the profit of firm \(i\) decreases in \(\theta_i\). Lastly, we have
\[
\gamma_{RP}^{i}(x, X, \theta_i) - \gamma_{RP}^{j}(x, X, \theta_j) = \frac{\theta_i - \theta_j}{n} \left[-P'(X) X + C'_i(x_i)\right] \leq 0,
\]
which implies that \(x_i \leq x_j\) for \(\theta_i \leq \theta_j\), while the strict inequality follows when \(\theta_i < \theta_j\).

### 3.5 Labor Managed Oligopoly

Many authors investigate the interaction between profit-maximizing and labor-managed Cournot oligopolies, while Stewart (1992) considers a delegation game in a mixed oligopoly setting in which a labor managed firm and a private firm compete each other, while the managers of
both firms are concerned with profit or the surplus to workers and output. We consider a mixed oligopoly model whose objective function is expressed by a linear combination of profit and the surplus to workers:

$$\Pi^L_i (x_i, X, \theta_i) \equiv (1 - \theta_i) [P (X) x_i - C_i (x_i)] + \theta_i \frac{P (X) x_i - C_i (x_i)}{N(x_i)}.$$  \hfill (26)

The first-order condition for profit maximization is given by

$$\gamma^L_i (x_i, X, \theta_i) \equiv (1 - \theta_i) [P' (X) x_i + P (X) - C'_i (x_i)] + \theta_i \left\{ \frac{P (X) + P' (X) x_i - C'_i (x_i)}{N(x_i)} - \left[ \frac{P (X) - C'_i (x_i)}{N(x_i)} \right] N^\prime(x_i) \right\} \leq 0. \hfill (27)$$

We assume

$$\frac{\partial \gamma^L_i (x_i, X, \theta_i)}{\partial x_i} = (1 - \theta_i) [P' (X) - C''_i (x_i)] + \theta_i \left\{ \frac{[P' (X) - C''_i (x_i)] N_i (x_i) - [P (X) + P' (X) x_i - C'_i (x_i)] N'_i (x_i)}{[N_i (x_i)]^2} \right. \left. \frac{-C''_i (x_i) N'_i (x_i) + [P (X) - C'_i (x_i)] N''_i (x_i)}{[N_i (x_i)]^4} \right. \left. + \frac{[P (X) - C'_i (x_i)] N'_i (x_i) 2 N_i (x_i) N'_i (x_i)}{[N_i (x_i)]^4} \right\}$$

$$= (1 - \theta_i) [P' (X) - C''_i (x_i)]$$

$$+ \theta_i \left\{ \frac{[P' (X) - C''_i (x_i)] N_i (x_i) - [P (X) + P' (X) x_i - C'_i (x_i)] N'_i (x_i)}{[N_i (x_i)]^2} \right. \left. \frac{-C''_i (x_i) N'_i (x_i) + [P (X) - C'_i (x_i)] N''_i (x_i)}{[N_i (x_i)]^4} \right. \left. + \frac{[P (X) - C'_i (x_i)] N'_i (x_i) 2 N_i (x_i) N'_i (x_i)}{[N_i (x_i)]^4} \right\}.$$
which implies that
\[ (1 - \theta_i) [P'(X) - C''_i(x_i)] + \theta_i \left\{ \frac{[P'(X) - C''_i(x_i)] N_i(x_i) - [P(X) + P'(X) x_i - C'_i(x_i)] N'_i(x_i)}{[N_i(x_i)]^2} \right\} < 0, \tag{28} \]

and
\[
\begin{align*}
&x_i \frac{\partial \gamma^L_i(x_i, X, \theta_i)}{\partial \theta_i} + X \frac{\partial \gamma^L_i(x_i, X, \theta_i)}{\partial X} = x_i (1 - \theta_i) [P'(X) - C''_i(x_i)] + X (1 - \theta_i) [P''(X) x_i + P'(X)] + \frac{X \theta_i}{N_i(x_i)} \left\{ P'(X) \left[ 1 - \frac{x_i N'_i(x_i)}{N_i(x_i)} \right] + P''(X) x_i \right\} < 0. \tag{29} \end{align*}
\]

We differentiate the RHS of (27) with respect to \( \theta_i \) to yield
\[
\frac{\partial \gamma^L_i(x_i, X, \theta_i)}{\partial \theta_i} = -[P'(X) x_i + P(X) - C'_i(x_i)] + \frac{P(X) + P'(X) x_i - C'_i(x_i)}{N(x_i)} - \frac{[P(X) - C'_i(x_i)] N'(x_i)}{[N(x_i)]^2},
\]
\[
= \frac{[-N(x_i) + 1] [P(X) + P'(X) x_i - C'_i(x_i)] - [P(X) - C'_i(x_i)] N'(x_i)}{[N(x_i)]^2} < 0, \tag{30} \]

which implies that \( \hat{X} \) increases. Furthermore,
\[
\frac{\partial \Pi^L_j(x_j, X, \theta_j)}{\partial X} = P'(X) x_j \left[ 1 - \theta_j + \frac{\theta_j}{N(x_j)} \right] < 0, \forall j \in I \text{ except for } i,
\]

which implies that \( \Pi^L_j(x_j, X, \theta_j) \) is strictly decreasing in \( X \) for all \( x_j \in (0, \omega_j] \), that is, the
profits of the firms other than $i$ fall in response to this change. Since

$$\Pi^L_i (0, X, \theta_i) = 0 = \Pi^L_i \left(0, X, \tilde{\theta}_i \right) \text{ for all } X,$$

the profit of firm $i$ decreases in $\theta_i$. On the other hand, we have

$$\gamma^L_i (x, X, \theta_i) - \gamma^L_j (x, X, \theta_j) = (\theta_j - \theta_i) \left[ P'(X) x_i + P(X) - C'_i (x_i) \right]$$

$$+ (\theta_i - \theta_j) \left\{ \frac{P(X) + P'(X) x_i - C'_i (x_i)}{N(x_i)} - \frac{[P(X) - C'_i (x_i)] N'(x_i)}{[N(x_i)]^2} \right\},$$

$$= (\theta_j - \theta_i) \left[ P'(X) x_i + P(X) - C'_i (x_i) \right.$$

$$- \frac{P(X) + P'(X) x_i - C'_i (x_i)}{N(x_i)} + \frac{[P(X) - C'_i (x_i)] N'(x_i)}{[N(x_i)]^2} \left\} \leq 0.\right.$$

That is, the output of firm $i$ may be larger or smaller than that of firm $j$.

**Remark 3** Kaneda and Matsui (2003) have assumed the conditions, such as $N'_i (x_i) > 0$, $N''_i (x_i) > 0$ and $N_i (x_i) - x_i N'_i (x_i) < 0$, to ensure the existence of a Cournot-Nash equilibrium. These assumptions hold under decreasing-returns-to-scale production function.\(^8\) In contrast, much weaker conditions is enough to prove the existence of a Nash equilibrium using the share function approach.

### 3.6 Social Welfare

In the literature on mixed oligopoly with a public firm, it is usually assumed that the government representative in firm $i$ cares only about the social welfare of the market which is given by the sum of the market-specific profit and consumer surplus. The objective function of firm (public firm) can be expressed by

$$\Pi^S_i (x_1, \ldots, x_n, X, \theta_i) \equiv (1 - \theta_i) \left[ P(X) x_i - C_i (x_i) \right] + \theta_i SW_i (x_1, \ldots, x_n, X), \quad (31)$$

----

\(^8\)This assumption amounts to that the production function of firm $i$ displays decreasing returns-to-scale. Indeed, denote such a production function as $x_i = f_i (N_i)$ with $f'_i (N_i) > 0$ and $f''_i (N_i) < 0$. We obtain $dN/dx = 1/f''(N_i) > 0$ and $d^2 N/dx^2 = -f''_i (N_i)/[f'(N_i)]^2 > 0$. 

20
where $\theta_i \in [0, 1]$ is the weight assigned on the objective other than profit and $SW(x_1, ..., x_n, X)$ represents consumer’s surplus defined precisely later.

Matsumura (1998) interprets the weight $\theta_i$ as a function of the share of public ownership of firm $i$, $s_i$, and thus $1 - s_i$ represents the share of private ownership of firm $i$; $s_i \in [0, 1]$. When $s_i = 1$, the oligopoly is a purely private firm, it cares only the amount of profits (i.e., $\theta_i(1) = 1$), while when $s_i = 1$, the oligopoly is a purely public firm, it cares only the amount of social welfare (i.e., $\theta_i(1) = 0$). In the intermediate case (i.e., $\theta_i \in (0, 1)$) the ownership is mixed, i.e., partial privatization. Since the ownership $1 - s_i$ gives rise to bargaining power of the private partner, a privatized firm maximizes a weighted average of the payoffs of the government and its own profit.

The social welfare $SW(., )$ is the sum of consumer’s surplus and profits by all firms in the market is given by

\[
SW(x_1, ..., x_n, X) \equiv \int_0^X P(q)dq - P(X)X + \sum_{j=1}^n \pi_j(x_j, X),
\]

\[
= \int_0^X P(q)dq - P(X)X + \sum_{j=1}^n [P(X)x_j - C_j(x_j)],
\]

\[
= \int_0^X P(q)dq - P(X)X + P(X)X - \sum_{j=1}^n C_j(x_j),
\]

\[
= \int_0^X P(q)dq - \sum_{j=1}^n C_j(x_j),
\]

\[
= CS(X) - CS(0) - \sum_{j=1}^n C_j(x_j),
\]

where $CS(q) \equiv \int P(q)dq$ and $CS'(q) = P(q) > 0$.

The first order condition for profit maximization is given by

\[
\gamma^S_i(x_i, X, \theta_i) \equiv (1 - \theta_i) [P'(X)x_i + P(X) - C'_i(x_i)] + \theta_i [P(X) - C'_i(x_i)] \leq 0, \tag{32}
\]

We assume
\[ \frac{\partial \gamma_i^S(x_i, X, \theta_i)}{\partial x_i} = (1 - \theta_i) [P'(X) - C''_i(x_i)] - \theta_i C''_i(x_i) < 0, \]  

(33) and

\[
x_i \frac{\partial \gamma_i^S(x_i, X, \theta_i)}{\partial x_i} + X \frac{\partial \gamma_i^S(x_i, X, \theta_i)}{\partial X} =
\]

\[
x_i \{ (1 - \theta_i) [P'(X) - C''_i(x_i)] - \theta_i C''_i(x_i) \} + X \{ (1 - \theta_i) [P''(X) x_i + P'(X)] + \theta_i P'(X) \},
\]

\[= x_i (1 - \theta_i) [P'(X) - C''_i(x_i)] + [XP'(X) - x_i C''_i(x_i)] \theta_i + X (1 - \theta_i) [P''(X) x_i + P'(X)] < 0.
\]

(34)

We differentiate (32) with respect to \( \theta_i \) to yield

\[
\frac{\partial \gamma_i^S(x_i, X, \theta_i)}{\partial \theta_i} = - [P'(X) x_i + P(X) - C'_i(x_i)] + [P(X) - C'_i(x_i)].
\]

\[
= - P'(X) x_i > 0.
\]

(35)

which implies that \( X \) increases. Furthermore, since it turns out that

\[
\frac{\partial \Pi_j^S(x_j, X, \theta_j)}{\partial X} = (1 - \theta_j) P'(X) - \theta_j P(X) < 0, \quad \forall \ j \in I \text{ except for } i,
\]

Pi_j^S(x_j, X, \theta_j) is strictly decreasing in \( X \) for all \( x_j \in (0, \omega_j] \); hence, the profits of the firms other than \( i \) fall in response to this change. Since

\[
\Pi_i^S(0, X, \theta_i) = CS(X) = \Pi_i^S(0, X, \tilde{\theta}_i) \text{ for all } X,
\]

the profit of firm \( i \) increases in \( \theta_i \). Lastly, we can show that

\[
\gamma_i^S(x, X, \theta_i) - \gamma_j^S(x, X, \theta_j) = (\theta_j - \theta_i) [P'(X) x + P(X) - C'_i(x)] + (\theta_i - \theta_j) [P(X) - C'_i(x)],
\]

\[= (\theta_j - \theta_i) [P'(X) x + P(X) - C'_i(x) - P(X) + C'_i(x)],
\]

\[= (\theta_j - \theta_i) P'(X) x \leq 0,
\]

which implies that \( x_i \leq x_j \) for \( \theta_i \leq \theta_j \), while the strict inequality follows when \( \theta_i < \theta_j \).
4 Stability

In this section, we investigate global stability of equilibria of the oligopoly game presented in the previous section. Stability’s assumption is needed to make comparative statics arguments economically meaningful in the sense that the system automatically moves to a new Nash equilibrium over time after some shock occurs since we have assumed myopic agents.

We focus on the Gradient Dynamics each firm increases the value of her strategy if and only if her marginal profit is positive. A motivation for Gradient Dynamics is that firms attempt to increase their profits by moving in the direction given by the gradient of their profits function.

\[ \dot{x}_i = \kappa_i \gamma_i^j (x_i, X, \theta_i) \quad \text{for all } i \in I, \]  

where \( \kappa_i > 0 \) denotes speed of response of \( i \), \( \gamma_i^j (x_i, X, \theta_i) \) represents the marginal profit associated with additional objective \( j \) and the initial point satisfies \((x_1(0), x_2(0), ..., x_n(0)) > 0\). Alternatively, the dynamics can be expressed by the Hahn’s best-reply dynamics:

\[ \dot{x}_i = \kappa_i [b_i (X_{-i}, \theta_i) - x_i] \quad \text{for all } i \in I, \]  

provided each oligopoly firm has a well-defined best response function, \( b_i (X_{-i}, \theta_i) \). As shown below, the negative slope of the share function is not sufficient to demonstrate the stability of a Cournot-Nash equilibrium.

A.7 If \((x_i, X)\) satisfies \( 0 < x_i < X \) and \( \gamma_i (x_i, X) = 0 \), then

\[ \frac{\partial \gamma_i (x_i, X, \theta_i)}{\partial X} \leq 0. \]  

This assumption

**Proposition 5** Suppose \( \hat{x} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n) \) is a non-critical Cournot-Nash equilibrium of a smooth aggregate game in which A.1—A.3 and A.7 hold for all firms. There exists a unique Cournot-Nash equilibrium which is asymptotically stable under the adjustment rule (36) or
The proof is found in Corchón (2001). Unfortunately, the negative slope of the share function is not sufficient to demonstrate the stability of a Cournot-Nash equilibrium. Instead, the negative slope of the best response function suffices to ensure its stability. This condition ensures not only the existence and uniqueness of the Nash equilibrium, but also its global stability. In cases of Revenue and Output the so-called Hahn condition, is sufficient to ensure

1. Revenue: \( F^R_i (x_i, X) \equiv P(X)x_i \),
2. Output: \( F^O_i (x_i, X) \equiv x_i \),
3. Market share: \( F^M_i (x_i, X) \equiv x_i/X \),
4. Profit per worker: \( F^{LM}_i (x_i, X) \equiv \frac{P(X)x_i - C_i(x_i)}{N(x_i)} \),
5. Relative profit: \( F^{RP}_i (x_i, X) \equiv \pi_i(x_i, X) - \frac{1}{n} \sum_{j=1}^{n} \pi_j(x_i, X) \),
6. Social welfare: \( F^{SW}_i (x_i, X) \equiv \int_0^X P(q)dq - P(X)X + \sum_{j=1}^{n} \pi_j(x_j, X) \).

5 Multiple Objectives and Non-linear Managerial Contracts

In this section we consider the oligopoly firm which has multiple objectives more than two. For example, the public firm may have several objectives such as pure profit-maximization, social welfare, revenue, and so forth. The manager’s incentive contract is written as a weighted sum of those purposes. The true objective function of firm \( i \) confronted at the multiple non-profit maximizing objectives (\( m \) types of purposes including profit maximizing objective) is given by:

\[
\Pi_i (x_i, X, \Theta_i) \equiv \theta_{i1} [P(X)x_i - C_i(x_i)] + \sum_{j=1}^{m} \theta_{ij} F^{ij}_i (x_i, X), \quad i = 1, 2, ..., n,
\]

\( ^9 \)There is another way to model the behavior of myopic agents. Since each oligopoly firm has a well-defined best response function, \( b_i(X_{-i}, \theta_i) \), and the dynamics can be expressed by the Hahn’s best-reply dynamics:

\[
\dot{x}_i = \kappa_i [b_i(X_{-i}, \theta_i) - x_i] \quad \text{for all } i \in I.
\]
where the vector $\Theta_i \equiv (\theta_{i1}, \theta_{i2}, \ldots, \theta_{im}) \in \mathbb{R}_+^m$ represents the weights of firm $i$ assigned to a variety of objectives, including marginal and non-marginal ones, with $\sum_{j=1}^m \theta_{ij} = 1$ and $F_{ij}(x_i, X)$ represents the objective $j$ of firm $i$. The conditions to ensure the existence and uniqueness of a Cournot-Nash equilibrium of the oligopoly model with multiple objectives are given by A.2 and A.3 with appropriate modifications of (4) and (5).

Another interesting extension is to consider a mixed oligopoly model in which several firms have different objectives; for example, the true objective function of one firm has the weighted average of profit and sale revenues, while other firms may have the weighted average of various components such as profit and output, profit and market share, social welfare and relative profit, the surplus to workers and output, and so on. In this case, as long as the marginal profit function of each oligopoly firm satisfies A.1-A.4, the existence and uniqueness of a Cournot-Nash equilibrium of the oligopoly competition can easily be guaranteed.

Although we have focused on linear managerial contracts so far, we may consider a more general non-linear managerial incentive contract such as

$$\Pi_i(x_i, X) \equiv B_i \left[ F^{i0}(x_i, X) , F^{i1}(x_i, X) , \ldots , F^{im}(x_i, X) \right] , \ i = 1, 2, \ldots, n,$$

where $F^{i0}(\cdot)$ represents the pure profit purpose, while $F^{ij}(x_i, X)$ represents the non-profit purposes. Since the manager receive a bonus according to this non-linear contract, that firms’ managers maximizes the above objective function. As before, as long as the marginal profit function of each oligopoly firm satisfies, A.2 and A.3, the existence and uniqueness of a Cournot-Nash equilibrium of the oligopoly competition are ensured.

6 Concluding Remarks

In this paper we investigate the existence, uniqueness and stability of Cournot oligopoly models with alternative non-profit objectives. We have shown that either Hahn’s condition or the generalized Hahn’s condition is not needed to ensure the existence, uniqueness and comparative statics properties in the oligopoly model with alternative non-profit maximizing objectives, at least, in the literatures on delegation game and mixed oligopoly. In other words, the Hahn’s
condition is neither sufficient nor necessary for the existence, uniqueness and stability for a Cournot-Nash equilibrium.

On the other hand, if we are only interested in the existence and uniqueness of the Nash equilibrium, assumptions A.2 and A.3, which together ensure the negative slope of the share function, are sufficient to prove them. These assumptions are much more weaker than Hahn's condition. The share function approach would provide the same general conditions to ensure the existence and uniqueness of the Cournot-Nash equilibrium irrespective of the form of the objective functions of oligopoly firms. These general conditions are useful to identify more specific sufficient conditions for the existence and uniqueness of the equilibrium.

The model and approach presented in this paper should be developed further in several directions. First of all, introducing product differentiation makes the present oligopoly model and results richer and more fruitful. We are practically interested in what form of utility functions generates a form of aggregate games. To conduct this extension, we have to apply the generalized aggregate game developed by Cornes and Hartley (2009), which will be left as our future work. Another interesting research agenda is to investigate the effect of changes in the tax, such as, commodity tax or the corporate income tax on the market equilibrium price and output. In particular, comparison of specific and ad valorem taxes would be an interesting research agenda. The aggregate game approach certainly provides a powerful tool to analyze these effects in a systematic way. The more challenging research agenda is to allow for some firms whose share functions are not monotonically decreasing. Nevertheless, as the number of firms which has a monotonic declining slope of the share function increases, such non-monotonic effect will be neutralized so that the aggregate share function would have a unique intersection with 45 degree line.
Appendix: Proof of Proposition 5

We will show that $V(x) = (1/2) \sum_{j=1}^{n} (x_j)^2$ is a Liapunov function. We set $\kappa_j = 1$ for $\forall i \in I$.

To this end, differentiation of the resulting expression with respect to time yields

$$
\dot{V}(x) = \sum_{j \in I} \left\{ (\dot{x}_j)^2 \left[ \frac{\partial \gamma_j(.)}{\partial x_j} + \frac{\partial \gamma_j(.)}{\partial X} \right] + \dot{x}_j \sum_{k \neq j} \dot{x}_k \frac{\partial \gamma_k(.)}{\partial X} \right\}. 
$$

(A.1)

A.2 and A.3 imply that all terms multiplying $(\dot{x}_i)^2$ are negative (due to A.2 and A.3), while the sign of the second term in the braces is ambiguous. If the second term is negative, we are done. We consider the case where the second term is positive; in particular, we consider the following two worst possible cases. (i) For $\forall j \in I$, $\dot{x}_j \sum_{k \neq j} \dot{x}_k < 0$ and $\partial \gamma_k(.) / \partial X < 0$ and (ii) for $\forall j \in I$, $\dot{x}_j \sum_{k \neq j} \dot{x}_k \geq 0$ and $\partial \gamma_k(.) / \partial X \geq 0$. In case (i)

$$
\dot{x}_j \sum_{k \neq j} \dot{x}_k \cdot \frac{\partial \gamma_k(.)}{\partial X} < \dot{x}_j \sum_{k \neq j} \dot{x}_k \left[ \frac{\partial \gamma_k(.)}{\partial x_k} + \frac{\partial \gamma_k(.)}{\partial X} \right],
$$

which implies

$$
\dot{V}(x) < \sum_{j \in I} \kappa_j \left[ (\dot{x}_j)^2 + \dot{x}_j \sum_{k \neq j} \dot{x}_k \right] \left[ \frac{\partial \gamma_k(.)}{\partial x_k} + \frac{\partial \gamma_k(.)}{\partial X} \right] < 0.
$$

There exists a constant $v$ such that

$$
v < \frac{\partial \gamma_k(.)}{\partial x_k} + \frac{\partial \gamma_k(.)}{\partial X} < 0 \text{ for } \forall k \in I.
$$

Hence,

$$
\dot{V}(x) < v \sum_{j \in I} \left[ (\dot{x}_j)^2 + \dot{x}_j \sum_{k \neq j} \dot{x}_k \right] = v \left( \sum_{j \in I} \dot{x}_j \right)^2 < 0,
$$

and $\dot{V}(x) = 0$ if only if $\dot{x}_j = 0$ for $\forall i \in I$.

In case (ii), on the other hand, we have

$$
\dot{x}_j \sum_{k \neq j} \dot{x}_k \cdot \frac{\partial \gamma_k(.)}{\partial X} > 0 > \dot{x}_j \sum_{k \neq j} \dot{x}_k \left[ \frac{\partial \gamma_k(.)}{\partial x_k} + \frac{\partial \gamma_k(.)}{\partial X} \right].
$$
which implies
\[ \dot{V}(x) < \sum_{j \in I} \left[ (\dot{x}_j)^2 + \dot{x}_j \sum_{k \neq j} \ddot{x}_k \right] \left[ \frac{\partial \gamma_k(\cdot)}{\partial x_k} + \frac{\partial \gamma_k(\cdot)}{\partial X} \right] < 0. \]

Taken together, \( V(x) \) is a Liapunov function.

References


