Capital accumulation, population, and taxation in an intergenerational model with Millian optimality

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Abstract

In this paper we study capital accumulation (via bequests), population, and taxation in an intergenerational model with Millian optimality and time-intensive child-rearing. We investigate conditions for social optima by overcoming some limitations in existing models due to the non-convexity in the feasible set caused by the tradeoff between investment and the number of children. The model displays local convergence at the steady state and allows for the possibility of the recent reversal of fertility declines in highly developed economies. The model also sheds new light on taxation. Higher income taxes not only reduce investment but also increase fertility. Higher consumption taxes at a time-invariant rate have similar effects but higher government debt has opposite effects. To finance committed lump-sum transfers, it is socially optimal to tax capital income and subsidize new investment at the same constant rate, or to tax labor income at an increasing and convergent rate, or to tax consumption spending at a constant rate, and to use government debt to offset the tax effects on fertility.

Keywords: Taxation; Government debt; Capital accumulation; Fertility; Bequests

JEL classification: D9; H0; J0; O4

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1. Introduction

Capital accumulation and fertility, commonly regarded as fundamental factors for the future, interact closely and vary significantly in the development process. In early development with low capital intensity, birth rates are high and saving rates are low. The 20th century has observed the completion of the Demographic Transition (the fertility declines) and the Industrialization (via capital accumulation) in today’s developed economies marked by low fertility rates and high capital intensity. A recent reversal of fertility declines is documented empirically in Myrskylä, Kohler and Billari (2009) for most advanced economies, casting doubt about a simple negative relation between fertility and capital intensity and motivating our investigation for explanations. Even among the G-7 countries, there are large variations in household savings and fertility rates, which are likely associated with the different policies on taxes and government debt in these most advanced countries (see Table 1). One may ask why those with much higher government debt over GDP (Italy and Japan) have much lower fertility rates and much higher private saving rates than the United States despite that the United States has lower tax revenue over GDP.

The standard theory of capital accumulation assumes an infinite horizon and exogenous fertility and predicts global convergence towards a unique steady state. It also links the social optimum to a competitive equilibrium and serves as the basis for the standard theory of taxation, known as the Ramsey taxation. This tax theory predicts a negative effect of future capital income taxes on investment and argues for high taxes on capital (or on its income) initially and for zero future capital income taxes according to Chamley (1986) and Judd (1985). As is well known, any commitment on such a tax policy lacks time consistency. High wage income taxes also have a negative effect on labor, investment and welfare as shown in Prescott (2002) with endogenous leisure. However, taxes on capital income and wage income are used in many countries with fairly steady rates.

When new investment is subsidized, the optimal capital income tax rate can be positive
and equal to the investment subsidy rate in the long run in Abel (2007) and Zhang, Davies, Zeng and McDonald (2008) with elastic leisure. However, their argument for opposite tax rates on consumption spending and wage income for equal taxation on consumption and leisure is not observed in practice. Moreover, their optimal tax may not apply unless government spending is less than capital income minus investment.

The typical assumption of exogenous fertility in the standard theory for growth and taxation limits its usefulness given the large variations in fertility rates observed across countries and across times. In fact, there has been a growing interest in the reformulation of economic theory of endogenous fertility and various criteria for allocation efficiency. The existing studies in this literature vary by whether parents value the total or average utility of children in addition to the number of children. When the number and total utility of children enter parental preferences, consumption cannot be positive unless children are net financial burdens to parents (in the necessary conditions)—the costs of children exceed their discounted earnings, as shown in Becker and Barro (1988); however, they point out that this condition is hardly observed in modern economies. When the number and average utility of children enter parental preferences, one can characterize social optima under additional conditions such as inelastic fertility and inelastic consumption as shown in Razin and Ben-Zion (1975); without time cost for a child in their model, however, fertility has no upper bound and cannot be determined in the case of logarithmic utility. Such limitations arise from the nonconvexity in the feasible set caused by the tradeoff between investment per child and the number of children. This nonconvexity blurs the conditions for social optima and for convergence towards the steady state.

In this paper we study capital accumulation (via bequests), fertility, and taxation in

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1Substantial investment subsidies are used in the United States according to Gordon, Kalambokidis and Slemrod (2004) and Gordon, Kalambokidis, Rohaly and Slemrod (2004).

an intergenerational model with time-intensive child rearing and with parental altruism towards the number and average utility of children. Facing a non-convex feasible set, we investigate the conditions for the social optimum with Millian optimality, the responses of investment and fertility to rising capital intensity, conditions for convergence, the equilibrium effects of taxes and government debt on investment and fertility, and socially optimal government policies. We make the following contributions.

The first contribution is to find conditions for the social optimum in this model that extends the literature with Millian optimality to incorporate a fixed time cost of rearing a child and partial capital depreciation (e.g. Razin and Ben-Zion, 1975; Lapan and Enders, 1990; van Groezen, Leers and Meijdam, 2003; Conde-Ruiz, Giménez, and Pérez-Nievas, 2010). The fixed time cost for a child sets an upper bound on fertility for a proper definition of the optimization problem. These extended elements also matter significantly for the dynamics and for the application to taxation. With the non-convexity in the feasible set, we make a precise assumption for an overall concave programming that is also useful for the analysis of dynamics and taxes. The extended model overcomes the aforementioned limitations in existing models, such as the possible unboundedness of fertility and the empirically implausible restriction on children as net financial burdens for positive consumption.

The second contribution is to determine the dynamic properties and to find conditions for convergence. On the dynamic path, fertility and investment are determined recursively by available capital (state dependency) in each period. With additively separable utility between consumption and fertility, capital is increasing in itself over time towards a steady state that is at least locally stable if higher capital intensity does not increase the marginal utility cost of a child at the steady state. The response of fertility to rising capital intensity is more likely to be positive (negative) if the cost of a child is higher (lower) and the time cost of a child is less (more) responsive to higher capital intensity. Starting from low capital intensity, the cost of a child (forgone wage and bequests) is low and the time cost of a child
rises sensitively to higher capital intensity. In this low capital intensity situation, fertility is likely to be high and to decrease with rising capital intensity. Under the same condition for convergence, fertility may respond positively to higher capital intensity at the steady state for plausible parameterizations. The model helps to explain the fertility declines in many countries and the recent reversal in fertility declines in most advanced economies.

The third contribution is to find the substitution effects of income taxes, consumption taxes and government debt on investment and fertility with additively separable utility between consumption and fertility. Investment and fertility are determined by available assets and government policies in each period in a recursive equilibrium. Higher income taxes in our model not only reduce investment but also increase fertility. Higher consumption taxes at a time-invariant rate have similar effects but higher government debt has opposite effects. These results help to explain why Italy with high taxes yet much higher government debt has low fertility but high private saving rates among the G-7 countries.

The fourth contribution is on the normative side of taxation by Millian optimality. To finance committed lump-sum transfers, it is socially optimal to tax capital income and subsidize new investment at the same constant rate, to tax labor incomes at an increasing and convergent rate, and to use government debt to offset the tax effects on fertility. The converge occurs under a plausible condition that children’s discounted earnings exceed their costs. Regardless of this condition, it is also socially optimal to tax consumption spending at a constant rate over time and to use government debt to offset the tax effect on fertility. Any mix of the two types of policy is socially optimal as well. The results are in line with the observed positive levels of taxes on capital and labor incomes and on consumption spending, investment subsidies and government debt.

The remainder of the paper proceeds as follows. Section 2 introduces the model. Section 3 focuses on the social optimum. Section 4 analyzes the equilibrium and government policy. The last section concludes. Technical proofs are relegated to the appendix.
2. The model

The model has discrete time that runs from period 0 to infinity, $t = 0, 1, ..., \infty$. In every period, the economy is inhabited by a large number of identical adult agents who die at the end of that period. With one unit of time endowment an adult agent chooses $N_t \in \mathbb{R}_+$ identical children and gives a bequest $K_{t+1} \in \mathbb{R}_+$ (capital) to each child. Rearing a child requires $v \in (0, 1)$ fixed units of time, setting an upper bound on fertility $1/v \geq N_t$, $\forall t \geq 0$. The remaining time is used to work: $L_t = 1 - vN_t$ which is decreasing in fertility. Children do not make any decision.

There is a single consumption-investment good. Feasibility in the economy is given by

$$C_t + N_t K_{t+1} - (1 - \delta)K_t = F(K_t, L_t),$$

where $C_t \in \mathbb{R}_+$ is consumption per worker, $K_t \in \mathbb{R}_+$ is available capital per worker (capital intensity), and $\delta \in (0, 1]$ is the rate of capital depreciation; $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is strictly increasing, strictly concave, twice differentiable, and homogeneous of degree one (constant returns to scale). Capital and labor are essential inputs such that $F(0, L_t) = F(K_t, 0) = 0$. The product $N_t K_{t+1}$ introduces non-convexity into the feasible set when $N_t$ and $K_{t+1}$ are chosen jointly in this model.

The preferences of a consumer-worker are of the Millian type (Mill, 1848):  

$$\sum_{t=0}^{\infty} \beta^t U(C_t, N_t), \quad 0 < \beta < 1,$$

where $\beta$ is a discount factor; $U : \mathbb{R}_+^2 \to \mathbb{R}$ is strictly increasing, strictly concave, and twice differentiable. We also assume that the production and utility functions satisfy Inada conditions so as to focus on an interior optimal allocation.

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3 According to Kotlikoff and Summers (1981), intergenerational transfers account for the bulk of capital accumulation in the United States.

4 The dramatic decline in fertility in industrial nations since the early 1970s has gone in tandem with the significant increase in female labor force participation rates, suggesting a strong tradeoff between labor supply and fertility captured in our model. For example, fertility in the US fell from 3.449 in the early 1960s to 2.056 in 2000. Meanwhile, the married female labor force participation rate rose from 30.5% to 61.1% and the total participation rate from 59.2% to 67.1% (US Census Bureau, various years).
With a non-convex feasible set, we need a further restriction on the utility function such that the problem of maximizing utility in (2) subject to feasibility in (1) and to $L_t = 1 - vN_t$ is strictly concave. Note that the problem may be reformulated as

$$V(K_0) = \max_{(K_{t+1}, N_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(F(K_t, 1 - vN_t) + (1 - \delta)K_t - N_tK_{t+1}, N_t),$$

by substituting these constraints into $U(C_t, N_t)$ and $F(K_t, L_t)$ for $C_t$ and $L_t$. Thus, the exact restriction for a concave programming problem is given below:

**Assumption 1.** The return function $W(K_t, K_{t+1}, N_t) \equiv U(F(K_t, 1 - vN_t) + (1 - \delta)K_t - N_tK_{t+1}, N_t)$ is strictly concave in $(K_t, K_{t+1}, N_t)$.

Precisely, this corresponds to a negative definite Hessian matrix of $W(K_t, K_{t+1}, N_t)$. Intuitively, starting from an equality between the marginal rate of substitution and the marginal rate of transformation (concerning $N_t$ and $K_{t+1}$), the former rate has to change faster than the latter rate for an interior optimum as illustrated in Razin and Ben-Zion (1975, Figure 1).

Our analysis proceeds in two stages. We analyze the social optimum first and then the competitive equilibrium and government policy.

### 3. The social optimum

A social planner’s problem is to maximize utility in (2)

$$V(K_0) = \max_{(C_t, K_{t+1}, N_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$$

subject to feasibility (1) and $L_t = 1 - vN_t$, given an initial capital stock $K_0 > 0$.

One advantage of the Millian approach is to write (3) in a recursive structure in the analysis of the dynamic properties (and taxes later). Denote $\Gamma(K_t) \equiv F(K_t, 1) + (1 - \delta)K_t$ as the upper bound on $K_{t+1}$ and define a functional equation below

$$V(K_t) = \max_{K_{t+1} \in [0, \Gamma(K_t)],[N_t \in [0, 1/v]} \{W(K_t, K_{t+1}, N_t) + \beta V(K_{t+1})\}. \quad (4)$$
The unknown function $V(K)$ is expected to be differentiable under our assumptions.

Let us denote marginal utility as $U_x(t) \equiv \partial U(C_t, N_t)/\partial X_t$, $X = C, N$. Under the Inada conditions for interior solutions, the first-order conditions are as follows (for $t = 0, 1, \ldots$):

$$K_{t+1} > 0 : \quad N_t U_c(t) = \beta V_k(t + 1) = \beta U_c(t + 1)[F_k(t + 1) + 1 - \delta], \quad (5)$$

$$N_t > 0 : \quad U_n(t) = U_c(t)[F_l(t)v + K_{t+1}]. \quad (6)$$

The transversality condition is

$$\lim_{t \to \infty} \beta t U_c(t)[1 + F_k(t) - \delta]K_t = 0. \quad (7)$$

Condition (5) is the intergenerational optimal condition via bequests, analogous to the typical intertemporal allocation of consumption via savings that has been at the center of the standard theory on growth and on taxes. It means that the loss in marginal utility from leaving one unit of bequests to each child should be equal to the gain in marginal utility per child from enhancing per child consumption through the marginal product of capital $F_k$ less the rate of capital depreciation $\delta$. In other words, the marginal rate of substitution between per child consumption and parental consumption $U_c(t)/U_c(t + 1)$ should be equal to the discounted return on investment per child $\beta[1 + F_k(t + 1) - \delta]/N_t$.

Condition (6) means that the gain in marginal utility from having a child should be equal to the loss in marginal utility from giving up parental consumption through leaving a bequest $K_{t+1}$ to the child as well as forgoing output $F_l(t)v$ (the bequest cost and the time cost of a child). In other words, the marginal rate of substitution between parental consumption and fertility $U_n(t)/U_c(t)$ should equal the marginal cost of a child $F_l(t)v + K_{t+1}$.

We establish the sufficiency of the first-order conditions and the transversality condition (under our assumptions) for the social optimum below.

**Proposition 1.** An allocation $\{C_{SP}^t, N_{SP}^t, K_{t+1}^{SP}\}_{t=0}^\infty$ is socially optimal if it satisfies the feasibility in (1), the optimal conditions in (5) and (6), the transversality condition in (7), and Assumption 1 for $t = 0, 1, \ldots$, given an initial capital stock per worker $K_0 > 0$.  

The sufficiency of these conditions for the social optimum is nontrivial for this kind of model with an infinite horizon given the nonconvexity in the feasible set caused by the tradeoff between investment and fertility. It lends us an exact restriction that will be useful for our analysis of the dynamic properties of the model and the effects of taxes and government debt. In fact, most related studies characterize the social optimum by the intertemporal and intratemporal first-order conditions without taking the nonconvexity issue and the transversality condition into account (e.g., Lapan and Enders, 1990; van Groezen, Leers and Meijdam, 2003). Some related studies consider the nonconvexity issue but not the transversality condition for the social optimum (e.g., Razin and Ben-Zion, 1975; Conde-Ruiz, Giménez and Pérez-Nievas, 2010).

3.1 Dynamics and stability

Equations (1), (5) and (6) lead to

\[
N_t U_c(F(K_t, 1 - vN_t) + (1 - \delta)K_t - N_tK_{t+1}, N_t) = \beta V_k(K_{t+1}),
\]

\[
U_n(F(K_t, 1 - vN_t) + (1 - \delta)K_t - N_tK_{t+1}, N_t) = U_c(F(K_t, 1 - vN_t) +
(1 - \delta)K_t - N_tK_{t+1}, N_t)[vF_l(K_t, 1 - vN_t) + K_{t+1}].
\]

These two equations determine \(K_{t+1}(K_t)\) and \(N_t(K_t)\) as functions of \(K_t\) recursively in each period despite that \(V(K_{t+1})\) is an unknown function. Such a state-dependent evolution of capital and fertility allows us to conduct the dynamic analysis below.

Differentiating (8) and (9) with respect to \(K_t\) yields:

\[
A = \begin{bmatrix}
\frac{dK_{t+1}}{dK_t} & \frac{dN_t}{dK_t} \\
\frac{dU_c}{dK_t} & \frac{dU_c}{dN_t}
\end{bmatrix}
= \begin{bmatrix}
N_t U_{cc}(t)[1 + F_k(t) - \delta] \\
U_{cc}(t)[vF_l(t) + K_{t+1}] [1 + F_k(t) - \delta] + U_c(t) vF_{lk}(t)
\end{bmatrix}
\]

where

\[
A = \begin{bmatrix}
U_{cc}(t)N_t^2 + \beta V_{kk}(K_{t+1}) & U_{cc}(t)N_t(F_l(t)v + K_{t+1}) - U_c(t) \\
U_{cc}(t)N_t(F_l(t)v + K_{t+1}) - U_c(t) & U_{nn}(t) + U_{cc}(t)(F_l(t)v + K_{t+1})^2 + U_c(t) F_{lt}(t)v^2
\end{bmatrix}
\]
with \( V_{kk}(t+1) = F_{kk}(t+1)U_c(t+1) + [1 + F_k(t+1) - \delta]^2U_{cc}(t+1) < 0 \). In so doing, we presume that \( V(K) \) is twice continuously differentiable. Actually, \( V(K) \) has only been shown in the literature to be continuously differentiable and concave; see, e.g., Stokey and Lucas (with Prescott, 1989). Having this in mind, one may interpret the expressions as finite differences instead of derivatives.

The elements of the matrix \( A \) are second derivatives of choice variables \((K_{t+1}, N_t)\), indicating the impacts of changing a choice variable on the net marginal benefits of \( K_{t+1} \) and \( N_t \) respectively. All of these elements are negative as expected. The elements in the matrix on the right-hand side of (10) are the impacts of changing the available capital stock \( K_t \) on the marginal utility costs of \( K_{t+1} \) and \( N_t \), respectively. The first element \( N_tU_{cc}(t)(1 + F_k(t) - \delta) \) of the right-hand matrix is a negative impact of a higher \( K_t \) on the marginal utility cost of \( K_{t+1} \), i.e. \( N_tU_c(t) \), tending to induce more investment and lower fertility.

The second element \( U_{cc}(t)[vF_l(t) + K_{t+1}][1 + F_k(t) - \delta] + U_c(t)vF_{lk}(t) \) of the right-hand matrix is the impact of a higher \( K_t \) on the marginal utility cost of a child, \( U_c(t)(vF_l(t) + K_{t+1}) \), with an ambiguous sign. The first term here captures a negative effect of a higher \( K_t \) on the marginal utility cost of a child \( U_c(t)(vF_l(t) + K_{t+1}) \) through a decline in the marginal utility caused by the marginal return \( 1 + F_k(t) - \delta \) on capital \( K_t \), tending to induce higher fertility and lower investment. The second term here captures a positive effect of a higher \( K_t \) on the marginal utility cost of the forgone output for a child \( U_c(t)vF_{lk}(t) > 0 \), tending to induce lower fertility and higher investment. If the first term is weaker than the second term such that \( U_{cc}(t)[vF_l(t) + K_{t+1}][1 + F_k(t) - \delta] + U_c(t)vF_{lk}(t) > 0 \) (a possible case at a small \( K_t \)), then a higher \( K_t \) tends to have a negative effect on fertility and an opposite effect on investment (a possible explanation to the Demographic Transition during the industrialization stage of development). The converse, however, may not be true because of the negative effect of a higher \( K_t \) on fertility via the decline in the marginal
utility cost of investment we mentioned above. Despite the large decline in fertility in the development process, there is also a recent reversal of fertility declines in advanced economies documented in Myrskylä, Kohler and Billari, 2009). To allow for rich fertility behavior, we assume

**Assumption 2.** *At the steady state* $U_{cc}(t)[vF_l(t) + K_{t+1}][1 + F_k(t) - \delta] + U_c(t)vF_{lk}(t) \leq 0.$

This essentially assumes that starting with higher capital intensity $K_t$ does not increase the marginal utility cost of a child at the steady state (for matured economies in late development). Assumption 2 is also consistent with inelastic consumption needed for a concave programming in this model (though to a different extent to be seen later). For a CRRA utility function (more details later), whether the restriction in Assumption 2 holds outside the steady state depends on the relative levels of the cost of a child $vF_l(t) + K_{t+1}$ and the response of the time cost of a child $vF_{lk}(t)$. Starting from lower capital intensity $K_t$, it is less likely to meet the restriction in Assumption 2 because the cost of a child $vF_l(t) + K_{t+1}$ should then be lower but the response of the time cost $vF_{kl}(t)$ should be stronger.

By the concavity of $W(K_t, K_{t+1}, N_t)$ under Assumption 1, the sign of $\det(A)$ is given below with additively separable utility for simplification:

**Lemma 1.** *With* $U_{cn}(t) = 0$, $\det(A) > 0$.

We can now obtain the responses of $K_{t+1}(K_t)$ and $N_t(K_t)$ to a change in $K_t$. Equation (10) and Lemma 1 lead to

$$
\frac{dK_{t+1}}{dK_t} = |A|^{-1}\det \left[ \frac{N_tU_{cc}(t)[1 + F_k(t) - \delta]}{U_{cc}(t)[vF_l(t) + K_{t+1}][1 + F_k(t) - \delta] + U_c(t)vF_{lk}(t)} \right]
$$

$$
N_tU_{cc}(t)F_l(t)v + K_{t+1} - U_c(t)
$$

$$
U_{nn}(t) + U_{cc}(t)(F_l(t)v + K_{t+1})^2 + U_c(t)F_{ll}(t)v^2
$$

10
\[
N_t(U_{cc}(t)U_{nn}(t)[1 + F_k(t) - \delta] + N_tU_c(t)U_{cc}(t)v^2F_{kl}(t)[1 + F_k(t) - \delta]
\]

\[
+ U_c(t)U_{cc}(t)[vF_l(t) + K_{t+1}][1 + F_k(t) - \delta] - N_tU_c(t)U_{cc}(t)vF_{lk}(t)
\]

\[
[vF_l(t) + K_{t+1}] + U^2_c(t)vF_{lk}(t)\{1 + F_k(t) - \delta\}
\]

\[
> 0
\]

under the second-order condition \(W_{22}W_{33} - [W_{23}]^2 > 0\) implied by Assumption 1 and shown in Appendix B. This positive response of investment to higher capital intensity is standard with exogenous fertility and sets the direction for development. It remains valid here with endogenous fertility due to Assumption 1, despite the possibility that higher capital intensity \(K_t\) may reduce the marginal utility cost of a child before reaching the steady state.

Equation (10) also leads to

\[
\frac{dN_t}{dK_t} = |A|^{-1}\{N_tU_c(t)U_{cc}(t)[N_tvF_{kl}(t) + 1 + F_k(t) - \delta] + 
\]

\[
\beta V_{kk}(K_{t+1})U_{cc}(t)(vF_l(t) + K_{t+1})(1 + F_k(t) - \delta) + U_c(t)vF_{kl}(t)\}\. \tag{12}
\]

The sign of \(dN_t/dK_t\) is ambiguous in general. The first term in the numerator is negative, capturing the decline in the marginal utility cost of investment and of the increased forgone output for children caused by a higher \(K_t\). The second term has an ambiguous sign, depending on whether a higher \(K_t\) increases or decreases the marginal utility cost of a child. It is likely negative at low capital intensity \(K_t\) whereby higher capital intensity may increase the marginal utility cost of a child. But the second term is nonnegative at the steady state under Assumption 2.

A key factor signing the response of fertility to higher capital intensity (the current state or the development stage) is the cost per child \(vF_l(t) + K_{t+1}\). When the time cost of child rearing is low starting from low capital intensity, it is likely that fertility \(N_t\) responds negatively to higher capital intensity \(K_t\) in the development process. Another key factor is the response of the time cost of a child to higher capital intensity, \(vF_{kl}(t)\). When capital intensity \(K_t\) is low, this response of the time cost to higher capital intensity is stronger,
making it more likely for fertility to respond negatively to higher capital intensity. As a result, at lower capital intensity, it is more likely that fertility responds negatively to higher capital intensity in the development process (a possible explanation for the Demographic Transition). At very high capital intensity, the cost of a child $vF_l(t) + K_{t+1}$ should be high but the response of the marginal product of labor to a higher $K_t$ is likely to be low, making it possible for fertility to respond positively to higher capital intensity. Thus, the model allows for a theoretical possibility for fertility to be positively associated with the state (a comprehensive index of capital) $K_t$ when the capital intensity is high. This provides a possible explanation to why the reversal of fertility declines occurs in most advanced economies in Myrskylä et al. (2009).

The steady state, $(K^*, N^*)$, is characterized by

$$N^* = \beta[1 + F_k(K^*, 1 - vN^*) - \delta], \quad (13)$$

$$U_n(F(K^*, 1 - vN^*) + (1 - \delta - N^*)K^*, N^*) =$$

$$U_c(F(K^*, 1 - vN^*) + (1 - \delta - N^*)K^*, N^*)[vF_l(K^*, 1 - vN^*) + K^*]. \quad (14)$$

An increasing and concave $V(K_t)$ means $[V_k(K_t) - V_k(K_{t+1})](K_t - K_{t+1}) \leq 0$. The first-order condition $V_k(K_{t+1}) = \beta^{-1}U_c(t)N_t$ and the envelope condition $V_k(K_t) = U_c(t)[1 + F_k(K_t, 1 - vN_t) - \delta]$ yield $V_k(K_t) - V_k(K_{t+1}) = U_c(t)[1 + F_k(K_t, 1 - vN_t) - \delta - \beta^{-1}N_t]$. With $U_c(t) > 0$, we have:

$$[1 + F_k(K_t, 1 - vN_t(K_t)) - \delta - \beta^{-1}N_t(K_t)](K_t - K_{t+1}) \leq 0, \quad (15)$$

where $N_t(K_t)$ is determined in (8) and (9). At the steady state $K_t = K_{t+1} = K^*$ and $N_t = N^*$ in (13) and (14), $1 + F_k(K^*, 1 - vN^*) - \delta - \beta^{-1}N^* = 0$ in (13), satisfying (15).

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5The literature has offered several different explanations to the Demographic Transition: subsistence consumption and human capital investment (e.g., Becker, Murphy and Tamura, 1990); mortality declines and human capital investment (e.g., Ehrlich and Lui, 1991, Lagerlöf, 2003); gender gaps (e.g., Galor and Weil, 1996); technological changes (e.g., Kremer, 1993, Galor and Weil, 2000); structural changes from one with a fixed input to one with capital accumulation (in, e.g., Hansen and Prescott, 2002, Tamura, 2006), among others.
Also, $K_t$ cannot converge towards zero, because $K_t$ being arbitrarily near 0 leads to $F_k(t)$ approaching $\infty$ but $N_t \leq 1/v$, implying that $K_t - K_{t+1} < 0$ at low capital intensity $K_t$.

If fertility were fixed at some $\bar{N}$ as in the neoclassical growth model, then global convergence for $K_t \to K^*$ would be obvious: the tenancy of diminishing marginal product, $F_{kk} < 0$, implies that whenever $K_t$ exceeds its steady state $K^*$ at $1 + F_k(K^*, 1 - v\bar{N}) - \delta - \beta^{-1}\bar{N} = 0$, $1 + F_k(K_t, 1 - v\bar{N}) - \delta - \beta^{-1}\bar{N} < 0$ and thus $K_t - K_{t+1} > 0$, and vice versa, according to $[1 + F_k(K_t, 1 - v\bar{N}) - \delta - \beta^{-1}\bar{N}](K_t - K_{t+1}) \leq 0$. However, when $N_t$ is endogenous, the convergence requires more structure because when $F_k(t)$ falls with higher $K_t$, $N_t$ may fall at the same time as well. So whether the convergence starts or continues depends on the relative degree of the dependency of investment and fertility on the initial state.

**Proposition 2.** Suppose Assumptions 1 and 2 are valid. For $U_{cn}(t) = 0$, on the socially optimal path, investment and fertility are state dependent, $[K_{t+1}(K_t), N_t(K_t)]$, with $dK_{t+1}/dK_t > 0$; the dynamic path converges towards a stable steady state (at least locally).

In the proof of this proposition, the restriction in Assumption 2 is needed for convergence. This restriction has an alternative meaning

$$\partial^2 W(K_t, K_{t+1}, N_t) / \partial K_t \partial N_t = -U_{cn}(t)(vF_l(t) + K_{t+1})(1 + F_k(t) - \delta) - U_c(t)vF_{kl}(t) > 0$$

under $U_{cn}(t) = 0$. That is, with additively separable utility between consumption and fertility ($U_{cn} = 0$), the marginal contribution of a feasible rise in fertility to welfare is increasing with capital intensity at the steady state, when holding investment constant. This essentially excludes the possibility of a decline in fertility to be so rapidly that fertility is always lower than the discounted gross return on investment $N_t < \beta[1 - F_k(t) - \delta]$.

With a stable steady-state level of $C$, $N$ and $K$ (hence $U_c$), the intertemporal optimal condition implies $N_t < 1 + F_k(t) - \delta$ in the long run, that is, the population growth rate $N - 1$ must be lower than the rate of return on capital $F_k - \delta$ (the real interest rate
in a competitive economy) in the long run. This observation supports the necessity of
the transversality condition for sequences \( \{K_{t+1}, N_t\}_{t=0}^{\infty} \) that satisfy feasibility (1) and the
first-order conditions (5) and (6), or satisfy (8) and (9), and the convergence condition
towards the steady state. Using the intertemporal optimal condition for successive substitu-
tion, we can rewrite the transversality condition as \( \lim_{t \to \infty} \beta^t U_c(t)[1 + F_k(t) - \delta]K_t = \lim_{t \to \infty} U_c(0)N_0N_1 \cdots N_{t-1}K_t/(R_{k1}R_{k2} \cdots R_{kt-1}) = 0 \)
where \( R_k \equiv 1 + F_k - \delta \). A violation of
the transversality condition with a bounded and stable steady-state \( K^* \) would contradict
\( \lim_{t \to \infty} N_t/[1 + F_k(t) - \delta] = \beta < 1 \) in the steady state.\(^6\) A thorough investigation into this
is outside the scope of this paper.

The social optimum in our model with average child utility in the parental utility is in
the spirit of Millian optimality and similar to that in Razin and Ben-Zion (1975), Lapan
and Enders (1990), van Groezen, Leers and Meijdam (2003), and Conde-Ruiz, Giménez,
and Pérez-Nievas (2010). However, they assume full capital depreciation within each period
and no time cost of rearing a child and make no use of the transversality condition for the
social optimum. Without the time cost of a child, fertility may have no upper bound and
thus may have no interior solution for some functional forms like the logarithmic utility
function.

This social optimum in our model differs from the counterpart in Becker and Barro
(1988) and Barro and Becker (1989) that assume the total utility of all children in the
parental utility. The interior allocation with that approach requires the forgone output
for an additional child exceed the discounted marginal product of labor of the child in the
next period \( vF_l(t) > F_l(t+1)/[1 - \delta + F_k(t+1)] \) (using notations in our model). However,
this inequality for children to be net financial burdens does not seem to be observed in the
modern economies, as admitted in Becker and Barro (1988, page 9). In fact, for popularly
used functional specifications and parameterizations, our model can generate an interior

\(^6\) The necessity of the transversality condition is a difficult problem for dynamic optimization with an
infinite horizon. See, e.g., Kamihigashi (2001) and cited studies therein.
optimal allocation with \( vF_i(t) < F_i(t+1)/(1-\delta + F_k(t+1)) \).

### 3.2 Examples

Suppose that the utility function is additively separable with a CRRA specification:

\[
U(C_t, N_t) = C_t^{1-\sigma} + \rho N_t^{1-\theta}, \quad \rho > 0, \quad \sigma > 0, \quad \theta > 0.
\]

To meet the second-order condition or Assumption 1, \( \rho, \sigma \) and \( \theta \) need to be large enough. Also, suppose that the production function is Cobb-Douglas:

\[
F(K_t, 1-vN_t) = \hat{A}K_t^\alpha (1-vN_t)^{1-\alpha}.
\]

The transitional dynamic path of the neoclassical growth model with this CRRA utility function has no reduced-form solution even with exogenous fertility. Thus, we illustrate \( vF_i(t) < F_i(t+1)/(1-\delta + F_k(t+1)) \) and \( dN_t/dK_t > 0 \) as possible situations at the steady state numerically for plausible parameterizations \( \alpha = 0.3, \rho = 0.7, \delta = 0.25, \beta = 0.55, \theta = 6, \sigma = 2, \hat{A} = 1.7 \) and \( v = 0.2 \) with one period corresponding to 30 years.\(^7\)

The steady-state solution yields \( N = 1.0438, C = 0.877, 1-\delta + F_k = 1.898 \) (an annual rate of 2.2%), and \( v(1-\delta + F_k) = 0.38 \) with \( F_i(t+1)/F_i(t) = 1 \) in the steady state. Thus, \( vF_i(t) < F_i(t+1)/(1-\delta + F_k) \) in the steady state for the given parameterization. In fact, this holds for a much wider range of parameterizations, e.g. a smaller discounting factor, a higher depreciation rate, or more time for a child. Moreover, \( (dN_t/dK_t)|A = 0.215 \) in the steady state with \( (vF_i^* + K^S)/Y^S = 43.8\% \), which is a plausible ratio of per child cost to output per worker in advanced economies when \( vF_i \) covers things like child consumption.

---

\(^7\)When one period corresponds to 30 years, the discount factor \( \beta = 1.02^{-30} = 0.55 \) corresponds to an annual rate of time preference 2%, while the depreciation rate at 0.25 corresponds to an annual rate 1%. The capital’s share parameter at \( \alpha = 0.3 \) and the coefficient of relative risk aversion at \( \sigma = 2 \) are in their widely used ranges in the literature. Also, \( \theta = 6 \) is used for inelastic fertility and for a concave programming as argued in Appendix B (also see Razin and Ben-Zion, 1975, and Conde-Ruiz, Giménez, and Pérez-Nievas, 2010). The rest of the parameterization is assumed to generate plausible values for fertility and the ratio of investment to output.
and forgone output for child rearing.\footnote{In many steady state simulations we carried out with different parameterizations, Assumption 2 is met in all the cases. However, \(dN/dK\) may be negative when the ratio of per child cost to output per worker is lower.}

When \(\sigma = \theta = 1\), the utility function becomes logarithmic, a case in which there exists no optimal solution in Razin and Ben-Zion (1975, Footnote 12). With full capital depreciation \(\delta = 1\) within one period (as assumed in Razin and Ben-Zion, 1975) and with log utility, however, our model can have an interior optimal solution. In particular, fertility \(N\) and the proportional allocations of output to investment and consumption, \(s = K_{t+1}N_t/Y_t\) and \(1 - s = C_t/Y_t\) respectively, are expected to be time invariant in this special case. The problem at time 0 can now be written as

\[
V(K_0) = \max_{s \in [0,1], N \in [0,1/v]} \frac{1}{1 - \beta} \left\{ \frac{\alpha \beta}{1 - \alpha \beta} \left[ \ln s + \ln \hat{A} + (1 - \alpha) \ln(1 - vN) - \ln N \right] + \ln(1 - s) + \ln \hat{A} + \frac{\alpha (1 - \beta)}{1 - \alpha \beta} \ln K_0 + (1 - \alpha) \ln(1 - vN) + \rho \ln N \right\}.
\]

The welfare function \(V(K_0)\) is concave in \((s, N)\) if \(\rho \geq \alpha \beta/(1 - \alpha \beta)\) that guarantees that the Hessian matrix is negative semi-definite. The solution is \(s = \alpha \beta\) and

\[
N = \frac{\rho (1 - \alpha \beta) - \alpha \beta}{v[\rho (1 - \alpha \beta) - \alpha \beta + 1 - \alpha]}.
\]

Note that fertility is positive and bounded under the condition for a concave \(V(K_0)\). The existence of an optimal interior solution with log utility over consumption and over fertility differs from Razin and Ben-Zion (1975). The reason for the difference is that we consider a time cost of rearing a child that sets an upper bound on fertility. Should we allow the time cost of a child to approach zero \(v \to 0\), then our model would become the same as Razin and Ben-Zion (1975). When doing so, the solution for fertility given above would approach infinity. In that sense, the problem would not be well defined.

With logarithmic utility, the Cobb-Douglas production function and full capital depreciation, the optimal intertemporal condition in (5) in the steady state becomes \(F_k = N/\beta\).
From this and the solution for fertility, the condition $vF_k = vN/\beta < 1$ corresponds to $\rho(1 - \alpha\beta) - \alpha\beta < \beta[\rho(1 - \alpha\beta) - \alpha\beta + 1 - \alpha]$ or equivalently $\rho > \beta/(1 - \beta) > \alpha\beta/(1 - \alpha\beta)$ for $\alpha \in (0, 1)$. That is, the condition for $vF_k < 1$ always satisfies the condition for optimal fertility to exist (for concave utility), and the set of $\rho$ such that $\alpha\beta/(1 - \alpha\beta) < \rho < \beta/(1 - \beta)$ is non-empty.

We now analyze the competitive equilibrium and government policy.

4. Competitive equilibrium and government policy

In a competitive economy, a worker or consumer at time $t$ allocates after-tax income to his own consumption and bequests to children that consist of both capital $N_tK_{t+1}$ and one-period government bonds $N_tB_{t+1}$. The consumer budget constraint is given by $(1 + \tau_{ct})C_t + N_tK_{t+1} - (1 - \delta)K_t - s_{kt}(N_tK_{t+1} - K_t) + N_tB_{t+1} = R_{bt}B_t + r_tK_t - \tau_{kt}(r_t - \delta)K_t + (1 - \tau_{wt})w_t(1 - vN_t) + T_t$. Here, $\tau_{ct} \in \mathbb{R}$ is the tax rate on consumption spending; $\tau_{wt} < 1$ and $\tau_{kt} < 1$ are tax rates on labor income and capital income, respectively; $s_{kt} > -1$ is a subsidy rate on new investment in capital; $T_t \in \mathbb{R}$ is a lump-sum government transfer; $w_t \in \mathbb{R}_+$ and $r_t \in \mathbb{R}_+$ are wage and capital rental rates, respectively, depending on average capital and average labor; and $R_{bt} \in \mathbb{R}_+$ is the return on government bonds. Wage and capital rental rates are determined competitively according to

$$w_t = F_l(\bar{K}_t, \bar{L}_t) \equiv F_l(t),$$

$$r_t = F_k(\bar{K}_t, \bar{L}_t) \equiv F_k(t),$$

where $\bar{K}$ and $\bar{L}$ are the economy-wide average levels of $K$ and $L$ per worker, respectively.

For ease of notation, we denote the return on capital as $R_{kt} \equiv 1 - s_{kt} + (1 - \tau_{kt})(r_t - \delta)$ and rewrite the consumer budget constraint as

$$(1 + \tau_{ct})C_t + N_tK_{t+1}B_{t+1}N_t = R_{bt}B_t + R_{kt}K_t + (1 - \tau_{wt})w_t(1 - vN_t) + T_t. \quad (18)$$
Without uncertainty, capital and government bonds are perfect substitutes and their returns must be equal \( R_{bt} = R_{kt} / (1 - s_{kt-1}) \) so as to rule out arbitrage opportunities. Consumers also face no-Ponzi-game constraints

\[
\lim_{t \to \infty} B_t / \Pi_{t=1}^{t-1} R_{bt} \geq 0, \quad \lim_{t \to \infty} K_t / \Pi_{t=0}^{t-1} R_{bt} \geq 0.
\]

Accordingly, the government budget constraint per worker on average is

\[
N_t \bar{B}_{t+1} + \tau_{ct} \bar{C}_t + \tau_{wt} w_t (1 - v \bar{N}_t) + \tau_{kt} (r_t - \delta) \bar{K}_t = R_{bt} \bar{B}_t + \bar{T}_t + s_{kt} (\bar{N}_t \bar{K}_{t+1} - \bar{K}_t).
\]

(19)

Here, government revenue from issuing new bonds and from collecting taxes on consumption spending and on labor and capital incomes is equal to government spending on debt repayment, income transfers, and subsidies on new investment.\(^9\)

In the decentralized competitive economy with taxes and transfers, the decision making process consists of two parts. First, utility-maximizing consumers choose the number of children and allocate their income to own consumption and bequests to their children and profit-maximizing firms choose capital and labor, taking the government policy and prices as given. Second, knowing the consumer’s and the firm’s decisions as functions of government policy, the government chooses its debt and the rates of taxes and investment subsidies such that it can achieve the social optimum.

The firm’s decision is characterized in (16) and (17). The consumer’s problem is to maximize utility in (2) subject to the consumer budget constraint (18) and non-negative asset holding in the long run \( \lim_{t \to \infty} B_t / \Pi_{t=1}^{t-1} R_{bt} \geq 0 \) and \( \lim_{t \to \infty} K_t / \Pi_{t=0}^{t-1} R_{bt} \geq 0 \), choosing \( \{C_t, N_t, K_{t+1}, B_{t+1}\}_{t=0}^\infty \) and taking prices \( \{R_{bt}, R_{kt}, r_t, w_t\}_{t=0}^\infty \) and government policies \( \{\tau_{ct}, \tau_{wt}, \tau_{kt}, s_{kt}, \bar{T}_t, \bar{B}_{t+1}\}_{t=0}^\infty \) as given. Define a vector for the prices and the government variables \( P_t = (\tau_{ct}, \tau_{kt}, \tau_{wt}, s_{kt}, \bar{T}_t, \bar{B}_{t+1}, R_{bt}, R_{kt}, w_t) \) and \( P = (P_1, P_2, ...) \).

Define the net wealth of a consumer as \( D_t \equiv K_t + B_t - \bar{B}_t \) where \( \bar{B}_t \) is the share of the negative asset from public debt per consumer. Also, define a period-utility function on the

\(^9\)The U.S. tax system allows for immediate expensing of quality new investment from corporate income taxes; for more details, see Gordon, Kalambokidis and Slemrod (2004).
state variables and on fertility by using the budget constraint:

$$W(K_t, B_t, K_{t+1}, B_{t+1}, N_t; P_t) \equiv U((1/(1+\tau_{ct}))[R_{bt}B_t + R_{kt}K_t + (1-\tau_{wt})w_t(1-vN_t)]$$

$$+T_t - (1-s_{kt})N_tK_{t+1} - N_tB_{t+1}], N_t).$$

The programming is defined in the functional equation below:

$$V(K_t + B_t - \bar{B}_t; P_t) = \max_{K_{t+1} \in [0,Y_{dt}], B_{t+1} \in [0,Y_{dt}], N_t \in [0,1/v]} \{W(K_t, B_t, K_{t+1}, B_{t+1}, N_t; P_t) + \beta V(K_{t+1} + B_{t+1} - \bar{B}_{t+1}; P_{t+1})\}$$  \hspace{1cm} (20)

where $Y_{dt} \in \mathbb{R}_+$ can be regarded as the consumer’s disposable income.

The first-order conditions of the consumer’s problem are as follows (for $t = 0, 1, ...$):

$$B_{t+1} > 0 : \quad \frac{N_tU_c(t)}{1+\tau_{ct}} = \beta V_{B_{t+1}}(K_{t+1} + B_{t+1} - \bar{B}_{t+1}; P_{t+1}) = \beta \frac{U_c(t+1)R_{bt+1}}{1+\tau_{ct+1}},$$  \hspace{1cm} (21)

$$K_{t+1} > 0 : \quad \frac{N_t(1-s_{kt})U_c(t)}{1+\tau_{ct}} = \beta V_{K_{t+1}}(K_{t+1} + B_{t+1} - \bar{B}_{t+1}; P_{t+1}) = \beta \frac{U_c(t+1)R_{kt+1}}{1+\tau_{ct+1}},$$  \hspace{1cm} (22)

$$N_t > 0 : \quad U_n(t) = U_c(t)\frac{[(1-\tau_{wt})wTv + (1-s_{kt})K_{t+1} + B_{t+1}]}{1+\tau_{ct}}.$$  \hspace{1cm} (23)

Here, $R_{bt+1} = R_{kt+1}/(1-s_{kt}) = [1-s_{kt+1}+(1-\tau_{kt+1})(r_{t+1}-\delta)]/(1-s_{kt})$ and $V_B = V_K = V_D$.

The transversality conditions are

$$\lim_{t \to \infty} \beta^t U_c(t)R_{bt}B_t = 0, \quad \lim_{t \to \infty} \beta^t U_c(t)[R_{kt}/(1-s_{kt-1})]K_t = 0.$$  \hspace{1cm} (24)

For government policy to be dynamically feasible over all periods, government spending on the repayment of initial debt and on the present value of government transfers and investment subsidies must be paid by the present value of tax revenue:

$$R_{bt0}B_0 + \sum_{t=0}^{\infty} \Pi_j^t \left( \frac{N_j-1}{R_{bj}} \right) [T_t + s_{kt}(N_tK_{t+1} - K_t)] =$$

$$\sum_{t=0}^{\infty} \Pi_j^t \left( \frac{N_j-1}{R_{bj}} \right) [\tau_{ct}C_t + \tau_{wt}w_t(1-vN_t) + \tau_{kt}(r_t - \delta)\bar{K}_t].$$  \hspace{1cm} (25)
**Definition 1.** Given initial $K_0 > 0$ and $B_0 > 0$, a competitive equilibrium consists of sequences of allocations $\{C_t, K_{t+1}, L_t, N_t\}_{t=0}^{\infty}$, prices $\{r_t(\bar{K}_t, \bar{L}_t), w_t(\bar{K}_t, \bar{L}_t)\}_{t=0}^{\infty}$, and government policies $\{\tau_{ct}, \tau_{kt}, \tau_{wt}, \bar{B}_t, R_{bt}, s_{kt}, \bar{T}_t\}_{t=0}^{\infty}$ for $t = 0, 1, \ldots$ such that: taking prices and government policies as given consumers and firms optimize and their decisions are feasible, satisfying (16), (17), (18), and (21)-(24); the government budget constraints (19) and (25) are satisfied; the prices are determined such that all markets clear with $L_t = 1 - vN_t$ and $C_t + N_tK_{t+1} - (1 - \delta)K_t = F(K_t, L_t)$; there is no arbitrage: $R_{bt} = R_{kt}/(1 - s_{kt-1})$; there is consistency such that $B_t = \bar{B}_t$, $C_t = \bar{C}_t$, $N_t = \bar{N}_t$, $T_t = \bar{T}_t$, $K_t = \bar{K}_t$, and $L_t = \bar{L}_t$; the net wealth per consumer equals $D_t = K_t$ at all times ($t = 0, 1, \ldots$).

According to these equilibrium conditions, it is easy to verify that the competitive allocation is socially optimal without government intervention at all, and that the social optimum can be supported by competitive prices. Namely, the Welfare Theorems apply in this model. With government committed spending, taxes, investment subsidies and government debt create wedges in the marginal comparisons of consumer decisions. The equilibrium impacts of such wedges can be very different from those in the existing literature with exogenously fixed fertility. The difference emerges because of the trade-off between fertility $N_t$ and investments $K_{t+1}$ and $B_{t+1}$ for children. To see it more clearly, we start with the simple case with exogenous fertility and fixed labor supply.

### 4.1 Substitution effects of taxes and subsidies with fixed fertility

For analytical convenience, we focus on a departure from zero taxes, zero transfers, zero government debt, and zero subsidies (laissez faire). We also assume an equal increase in the amount of the proportional taxes and lump-sum transfers to focus on the substitution effects of any government policy change. This approach allows us to make use of Assumption 1 to sign the effects of each policy change more easily. For now, we assume that fertility is fixed at $N_t = \bar{N} > 0$ at all times as in the standard literature on taxation.
With fixed fertility, the equilibrium conditions (22), (18) and (19) and the consistency 
\((B = \bar{B},...)\) lead to

\[
\left[ \frac{\tilde{N}(1-s_{kt})}{1 + \tau_{ct}} \right] U_c \left( F(K_{t}, 1 - v\tilde{N}) + (1 - \delta)K_t - \tilde{N}K_{t+1}, \tilde{N} \right) = \beta V_k(K_{t+1}; P_{t+1})
\]

where

\[
V_k(K_{t+1}; P_{t+1}) = U_c \left( F(K_{t+1}, 1 - v\tilde{N}) + (1 - \delta)K_{t+1} - \tilde{N}K_{t+2}, \tilde{N} \right) \times
\]

\[
\left[ \frac{1 - s_{kt+1} + (1 - \tau_{kt+1})(F_k(K_{t+1}, 1 - v\tilde{N}) - \delta)}{1 + \tau_{ct+1}} \right]
\]

which determines \(K_{t+1}\) as a function of \(K_t\) and certain government policy in each pe-
riod. The equilibrium effects of government policy can be determined by differentiating the 
equilibrium law of motion (26). The signs of the effects can be determined by using the 
second-order condition and the transversality condition when needed.

As a reminder, note that the current capital income tax rate \(\tau_{kt}\) has no impact on 
current investment:

\[
\frac{dK_{t+1}}{d\tau_{kt}} = 0. \quad (27)
\]

This is because \(\tau_{kt}\) does not affect the future return on current investment at \(t\), explaining 
why initial taxes on capital or capital income are efficient.

By contrast, a rise in the capital income tax in the next period reduces the return to 
current investment, tending to discourage investment:

\[
\frac{dK_{t+1}}{d\tau_{kt+1}} = \frac{-\beta V_{k\tau_k}(t + 1)}{\beta V_{kk}(t + 1) + N^2U_{cc}(t)} < 0 \quad (28)
\]

since \(V_{kk}(t + 1) \equiv V_{K_{t}K_{t+1}}(K_{t+1}; P_{t+1}) = U_{cc}(t + 1)R_{kt+1}^2 + (1 - \tau_{kt+1})U_c(t + 1)F_{kk}(t + 1) < 0\) and \(V_{k\tau_k}(t + 1) \equiv V_{K_{t+1}\tau_{kt+1}}(K_{t+1}; P_{t+1}) = -(F_k(t + 1) - \delta)U_c(t + 1) < 0\). The result here 
supports the standard argument that a higher future capital income tax rate on the return
to capital impedes capital accumulation in equilibrium. This result justifies why future capital income should not be taxed in the Ramsay tax system.\footnote{The Ramsey tax result is found to be robust even when some of their stringent assumptions are relaxed as shown in a number of studies in the survey articles by Chari and Kehoe (1999) and Atkeson, Chari and Kehoe (1999). A recent example is Kocherlakota (2005) with private information about skills. Moreover, the capital income tax is typically regarded to be more distortionary than taxes on wage income and consumption spending with or without a tradeoff between labor and leisure; see, e.g., Summers (1981), Seidman (1984), Auerbach and Kotlikoff (1987), Chari, Christiano and Kehoe (1994), Devereux and Love (1994), Davies, Zeng and Zhang (2000), Turnovsky (2000), Auerbach and Hines (2002), and Lucas (2003)}

Conversely, an increase in the investment subsidy rate matched by a decline (rise) in lump-sum transfers (taxes) promotes capital accumulation (as intended by the government):

\[
\frac{dK_{t+1}}{ds_{kt}} = \frac{-\tilde{N}U_c(t)}{N^2U_{cc}(t) + \beta V_{kk}(t + 1)} > 0. \tag{29}
\]

To a lesser extent, an increase in a time-invariant subsidy rate

\[
\frac{dK_{t+1}}{ds_k} = \frac{-[\tilde{N}U_c(t) - \beta U_c(t + 1)]}{N^2U_{cc}(t) + \beta V_{kk}(t + 1)} > 0, \tag{30}
\]

where \(\tilde{N}U_c(t) = \beta U_c(t + 1)R_{kt+1}/(1 - s_k) > \beta U_c(t + 1)\). Here, \(R_{kt+1} > \tilde{N}(1 - s_k)\) in general because the intertemporal condition (22) and the transversality condition (24) imply

\[
U_c(0) \lim_{t \to \infty} \frac{K_{t+1}^{\tilde{N}t(1 - s_k)^t}}{R_k R_{k, t-1} \cdots R_{k, 1}} = 0.
\]

When \(s_{kt} = s_{kt+1} = \tau_{kt+1}\), their wedges are canceled out in the intertemporal optimal condition (so that \(V_{ks_k} = V_{ktk} = 0\)) and thus they have no effect on capital accumulation, a result that would be valid even with a leisure-labor tradeoff as in Abel (2007) and Zhang et al. (2008).

Because there is no wedge created by the labor income tax \(\tau_w\) in the intertemporal condition, the labor income tax has no impact on capital accumulation:

\[
\frac{dK_{t+1}}{d\tau_{wt}} = \frac{dK_{t+1}}{d\tau_{wt+1}} = 0. \tag{31}
\]
This result would not be valid should a tradeoff between leisure and labor be present: one would expect a positive effect of higher wage income taxes on leisure and a negative effect on investment as shown in Prescott (2002).

The impact of a higher current consumption tax on current investment is positive

\[
\frac{dK_{t+1}}{dτ_{ct}} = \frac{-\tilde{N}_t U_c(t)}{\tilde{N}_t U_{cc}(t) + \beta V_{kk}(t+1)} > 0.
\]

This is because a higher current consumption tax makes current consumption more expensive than future consumption. When \(τ_{ct} = τ_{ct+1} = τ_c\), however, a rise in the consumption tax rate across times has no effect on capital accumulation since it cancels out the wedges from both sides of the intertemporal condition. Also, a rise in \(B_{t+1}\) has no effect on the intertemporal conditions, which is well known as the Ricardian equivalence hypothesis with an infinite horizon as shown in Barro (1974) whereby government debt creates no net wealth when \(B_t = \bar{B}_t\) and \(D_t = K_t + B_t - \bar{B}_t = K_t\).

The results above reflect the conventional views about the effects of taxes, investment subsidies and government debt on capital accumulation with exogenous fertility. Several factors have been identified in the literature to explain why capital income taxes may be good even in the long run. Assuming uninsurable idiosyncratic risks, Aiyagari (1994) suggests that positive capital income taxes may be optimal to mitigate the overaccumulation of capital driven by precautionary saving. Even with complete insurance markets to remove idiosyncratic risks, Hubbard and Judd (1987) show that capital income taxes may be desirable in the presence of borrowing constraints. Moreover, life cycle considerations make savings inelastic and may thus justify positive capital income taxes according to Erosa and Gervais (2002) and Garriga (2003). Quantitatively, Conesa, Kitao and Krueger (2009) find a high optimal capital income tax rate and a progressive labor income tax when taking these factors into consideration. These factors are omitted in our frictionless, intergenerational model without uncertainty.
4.2 Substitution effects of taxes and subsidies with endogenous fertility

Again, we look at the departure from laissez faire and focus on an equal change in taxes and transfers for substitutions effects. In addition, we assume additively separable preferences between consumption and fertility, $U_{cn} = 0$, for simplicity.

With endogenous fertility, the equilibrium conditions (22), (18) and (19) and the consistency ($B = \bar{B}, ...$) lead to

$$\frac{N_t(1-s_{kt})}{1+\tau_{ct}} U_c(F(K_t, 1-vN_t)+(1-\delta)K_t-N_tK_{t+1}, N_t) = \beta V_k(K_{t+1}; P_{t+1})$$ (33)

where

$$V_k(K_{t+1}; P_{t+1}) = U_c(F(K_{t+1}, 1-vN_{t+1})+(1-\delta)K_{t+1}-N_{t+1}K_{t+2}, N_{t+1}) \times$$

$$\left[\frac{1-s_{kt+1}+(1-\tau_{kt+1})(F_k(K_{t+1}, 1-v\bar{N})-\delta)}{1+\tau_{ct+1}}\right].$$

There is also an equilibrium intratemporal optimal condition:

$$U_n(F(K_t, 1-vN_t)+(1-\delta)K_t-N_tK_{t+1}, N_t) = [F_i(K_t, 1-vN_t)+K_{t+1}] \times$$

$$U_c(F(K_t, 1-vN_t)+(1-\delta)K_t-N_tK_{t+1}, N_t).$$ (34)

Together, (33) and (34) form an equilibrium law of motion determining $N_t$ and $K_{t+1}$ recursively as functions of $K_t$ and government policy. The equilibrium analysis of the impacts of taxes and subsidies can now be based on (33) and (34) and Assumption 1.

Note that a change in $\tau_{kt}$ creates no wedges in the inter- and intra-temporal optimal conditions at time $t$ in (33) and (34). So the effect of $\tau_{kt+1}$ does not directly go through $N_{t+1}$ and $1-vN_{t+1}$. Differentiating the two equations with respect to $\tau_{kt+1}$ yields:

$$A \left[ \begin{array}{c} \frac{dK_{t+1}}{d\tau_{kt+1}} \\ \frac{dN_{t+1}}{d\tau_{kt+1}} \end{array} \right] = \left[ \begin{array}{c} -\beta V_{k\tau_k}(t+1) \\ 0 \end{array} \right]$$ (35)

where $|A| > 0$ from Lemma 1, $V_{kk}(t+1) = F_{kk}(t+1)U_c(t+1)+[1+F_k(t+1)-\delta]^2U_{cc}(t+1) < 0$ and $V_{k\tau_k}(t+1) = -[F_k(t+1)-\delta]U_c(t+1) < 0$.
Hence, we have
\[ \frac{dK_{t+1}}{d\tau_{kt+1}} = |A|^{-1} \begin{vmatrix} -\beta V_{kt}(t+1) & U_{cc}(t)N_t(f_t)v + K_{t+1} - U_c(t) \\ 0 & U_{nn}(t) + U_{cc}(t)(f_tv + K_{t+1})^2 + U_c(t)F_{tt}(t)v^2 \end{vmatrix} \]

\[ < 0, \]  

(36)

\[ \frac{dN_t}{d\tau_{kt+1}} = |A|^{-1} \begin{vmatrix} U_{cc}(t)N_t^2 + \beta V_{kk}(t+1) & -\beta V_{kt}(t + 1) \\ U_{cc}(t)N_t(f_tv + K_{t+1}) - U_c(t) & 0 \end{vmatrix} \]

\[ > 0, \]  

(37)

\[ \frac{dN_tK_{t+1}}{d\tau_{kt+1}} = -\beta V_{kt}(t+1)|A|^{-1}\{U_c(t)K_{t+1} + U_{nn}(t)N_t + U_{cc}(t)N_tvF_t(t)[vF_t(t) + K_{t+1}] + U_c(t)N_tF_{tt}(t)v^2\} \]

\[ < 0, \]  

(38)

because
\[ \{U_c(t)K_{t+1} + U_{nn}(t)N_t + U_{cc}(t)N_tvF_t(t)[vF_t(t) + K_{t+1}] + U_c(t)N_tF_{tt}(t)v^2\} \]

\[ < U_{nn}(t)N_t + U_c(t)N_tF_{tt}(t)v^2 + 2U_c(t)[vF_t(t) + K_{t+1}] \]

\[ < \frac{[U_c(t)]^2}{U_{cc}(t)N_t} < 0 \quad \text{(under } W_{22}(t)W_{33}(t) - [W_{23}(t)]^2 > 0 \text{ given in Appendix B).} \]

The effects of the subsidy rate \( s_{kt} \) on \( K_{t+1} \) and \( N_t \) are ambiguous when the subsidy is financed by a lump-sum tax (negative lump-sum transfers). However, the effect of the subsidy rate on the total capital stock next period is positive:
\[ \frac{dN_tK_{t+1}}{ds_{kt}} = -|A|^{-1}\{U_c(t)U_{nn}(t)N_t^2 + U_c(t)U_{cc}(t)[N_tF_t(t)v]^2 + [U_c(t)]^2F_{tt}(t)N_t^2v^2 + \]

\[ [U_c(t)]^2N_tK_{t+1} + \beta V_{kk}(t + 1)U_c(t)K_{t+1}^2 + N_tK_{t+1}[U_c(t)]^2\} \]

\[ > 0 \]  

(39)
using a similar argument as that signing $dN_t K_{t+1}/d\tau_{kt+1} < 0$.

Further, when $s_{kt} = s_{kt+1} = \tau_{kt+1} = \tau_k$, their direct effects on the intertemporal optimal condition cancel out but the negative effect of the subsidy on the bequest cost of a child remains, leading to

$$
\frac{dK_{t+1}}{d\tau_k} = |A|^{-1} \det \begin{bmatrix}
-1 & U_{cc}(t)N(t)F(t)v + K_{t+1} - U_c(t) \\
-U_c(t)K_{t+1} & U_{nn}(t) + U_{cc}(t)(F(t)v + K_{t+1})^2 + U_c(t)F(t)v^2
\end{bmatrix} < 0,
$$

(40)

$$
\frac{dN_t}{d\tau_k} = |A|^{-1} \det \begin{bmatrix}
-1 & U_{cc}(t)N_t^2 + \beta V_{kk}(t+1) \\
U_{cc}(t)N(t)F(t)v + K_{t+1} - U_c(t) & -U_c(t)K_{t+1}
\end{bmatrix} > 0,
$$

(41)

We state these results below:

**Proposition 3.** With $U_{cn}(t) = 0$ and starting at laissez faire, a rise in $\tau_{kt+1}$ for lump-sum transfers increases $N_t$ and reduces $K_{t+1}$ and $N_t K_{t+1}$. Also, a rise in $s_{kt}$ increases $N_t K_{t+1}$. Moreover, a rise in $\tau_{kt+1} = s_{kt} = s_{kt+1}$ increases $N_t$ and reduces $K_{t+1}$.

With endogenous fertility, a higher capital income tax reduces investment as in the case with exogenous fertility. However, the decline in investment per child reduces the bequest cost per child, thereby reducing the parent to have more children.

The positive effect of an investment subsidy on total investment accords with the standard view about the effect of investment subsidies on capital accumulation but has a richer interpretation with endogenous fertility. A rise in the investment subsidy $s_{kt}$ reduces the cost of investment and the cost of a child at the same time, tending to induce more investment and more children and leading to a rise in total investment $N_t K_{t+1}$. However, due to the tradeoff between investment per child and the number of children, one of $N_t$ and $K_{t+1}$ may decline.
In the case with $\tau_{kt+1} = s_{kt} = s_{kt+1}$, though their wedges in the intertemporal optimal condition cancel out, their negative effect on the bequest cost of a child leads to higher fertility. This increase in fertility reduces the after-tax return on investment per child and thus reduces investment per child, unlike in the case with exogenous fertility.

The effects of the wage income tax rate are

$$\frac{dK_{t+1}}{d\tau_{wt}} = |A|^{-1} \det \begin{bmatrix} 0 & N_t U_{cc}(t)[F_i(t)v + K_{t+1}] - U_c(t) \\ -U_c(t)F_i(t)v & \cdots \end{bmatrix}$$

$$< 0 ,$$

(42)

$$\frac{dN_t}{d\tau_{wt}} = |A|^{-1} \det \begin{bmatrix} U_{cc}(t)N_t^2 + \beta V_{kk}(t + 1) & 0 \\ \cdots & -U_c(t)F_i(t)v \end{bmatrix}$$

$$> 0 .$$

(43)

We state these effects of wage income taxes below.

**Proposition 4.** With $U_{cn}(t) = 0$ and starting at laissez faire, a rise in $\tau_{wt}$ for lump-sum transfers increases $N_t$ and reduces $K_{t+1}$.

A rise in the labor income tax rate reduces the after-tax return to labor (the opportunity cost of spending time rearing a child), tending to raise fertility. However, a rise in fertility reduces the return to investment per child in the intertemporal condition. Thus, the parent also reduces investment per child, which differs from the conventional view with fixed fertility. The negative effects of the wage income tax on investment and labor are similar to those with a leisure-labor tradeoff. But the mechanisms channeling the effects differ in these different models.

The effects of raising the consumption tax rate for lump-sum transfers are

$$\frac{dK_{t+1}}{d\tau_{ct}} = -|A|^{-1} \times$$

$$\det \begin{bmatrix} N_t U_c(t) & N_t U_{cc}(t)[F_i(t)v + K_{t+1}] - U_c(t) \\ U_c(t)[F_i(t)v + K_{t+1}] & U_{nn}(t) + U_{cc}(t)[F_i(t)v + K_{t+1}]^2 + U_c(t)F_{lt}(t)v^2 \end{bmatrix}$$
\[ dN_t \over d\tau_c = -|A|^{-1} \det \begin{bmatrix} \frac{U_{cc}(t)N_t^2 + \beta V_{kk}(t+1)}{N_t U_{cc}(t)[F_i(t)v + K_{t+1}] - U_c(t)} & U_{c}(t)[F_i(t)v + K_{t+1}] \\ U_c(t)[F_i(t)v + K_{t+1}] & \frac{N_t U_{cc}(t) + U_c(t)[F_i(t)v + K_{t+1}]}{2 + U_c(t)F_i(t)v^2} \end{bmatrix} > 0, \] (44)

which may be positive or negative, depending on the elasticity of intertemporal substitution and on the cost of a child.

When the consumption tax rate remains constant over time \( \tau_{ct} = \tau_c \), it cancels out from the intertemporal condition so that \( V_{k\tau_c} = 0 \); its effect on the cost of a child still exists:

\[ dK_{t+1} \over d\tau_c = -|A|^{-1} \times \]

\[ \det \begin{bmatrix} 0 & N_t U_{cc}(t)[F_i(t)v + K_{t+1}] - U_c(t) \\ U_c(t)[F_i(t)v + K_{t+1}] & \frac{N_t U_{cc}(t) + U_c(t)[F_i(t)v + K_{t+1}]}{2 + U_c(t)F_i(t)v^2} \end{bmatrix} < 0, \] (46)

\[ dN_t \over d\tau_c = -|A|^{-1} \det \begin{bmatrix} U_{cc}(t)N_t^2 + \beta V_{kk}(t+1) & 0 \\ N_t U_{cc}(t)[F_i(t)v + K_{t+1}] - U_c(t) & U_{c}(t)[F_i(t)v + K_{t+1}] \end{bmatrix} > 0. \] (47)

We thus have

**Proposition 5.** With \( U_{cn}(t) = 0 \) and starting at laissez faire, a rise in \( \tau_{ct} \) for lumpsum transfers increases \( K_{t+1} \) and has an ambiguous effect on \( N_t \). However, a rise in \( \tau_{ct} = \tau_{ct+1} = \tau_c \) reduces \( K_{t+1} \) and raises \( N_t \).

A rise in the current consumption tax rate reduces the cost of investment and the cost of a child at the same time, tending to raise investment per child and fertility. However, a
rise in investment per child also means a higher cost of a child, tending to reduce fertility. The net effect on fertility is unclear; it is more likely to be positive if investment is less elastic (large $-V_{kk}$ relative to $V_k$) or if the cost of a child is larger.

However, a rise in the same consumption tax rate currently and in the next period reduces the marginal benefit of investment to the same extent it reduces the cost of investment. Unlike in the case with exogenous fertility, the consumption tax is not neutral now, since it reduces the cost of a child and has a positive effect on fertility. The increase in fertility reduces the after-tax return to investment per child, leading to a decline in investment. Again, these results on investment and labor are similar to those with a labor-leisure tradeoff but work out through different mechanisms.

The effects of issuing more government bonds for lump-sum transfers today (serviced by lump-sum taxes next period) are

$$\frac{dK_{t+1}}{dB_{t+1}} = |A|^{-1} \det \begin{bmatrix} 0 & N_tU_{cc}(t)[F_l(t)v + K_{t+1}] - U_c(t) \\ U_c(t) & U_{nn}(t) + U_{cc}(t)[F_l(t)v + K_{t+1}]^2 + U_c(t)F_l(t)v^2 \end{bmatrix}$$

$$= -|A|^{-1}U_c(t)\{N_tU_{cc}(t)[F_l(t)v + K_{t+1}] - U_c(t)\}$$

$$> 0, \quad (48)$$

$$\frac{dN_t}{dB_{t+1}} = |A|^{-1} \det \begin{bmatrix} U_{cc}(t)N_t^2 + \beta V_{kk}(t + 1) & 0 \\ N_tU_{cc}(t)[F_l(t)v + K_{t+1}] - U_c(t) & U_c(t) \end{bmatrix}$$

$$= |A|U_c(t)\{U_{cc}(t)N_t^2 + \beta V_{kk}(t + 1)\}$$

$$< 0. \quad (49)$$

Here, $V_{KB}(t + 1) = 0$ because in equilibrium $V_K(t + 1) = U_c(t + 1)[1 + F_k(t + 1) - \delta]$ is not directly affected by $B_{t+1}$ with $C_t$ determined by feasibility in (1). These lead to

**Proposition 6.** With $U_{cn}(t) = 0$ and starting at laissez faire, a rise in government bonds $B_{t+1}$ for current lump-sum transfers (serviced by lump-sum taxes next period) increases $K_{t+1}$ and reduces $N_t$. 

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Through increasing the bequest cost of a child, a rise in the government debt for lump-sum transfers reduces fertility. The decline in fertility increases the return to investment per child, inducing a positive response of investment per child at the same time. Similar results about government debt or social security were obtained with endogenous fertility in the steady state (e.g., a series of work by Nerlove et al.; Lapan and Enders, 1990), on the balanced path (e.g., Zhang, 1995, 2003), in non-recursive life-cycle models (e.g., Eckstein and Wolpin, 1985; Wildasin, 1990), or in stylized models with simple functions (e.g., van Groezen, Leers and Meijdam, 2003; Zhang and Zhang, 2007; Yew and Zhang, 2009). The effects of government debt in our model apply to the entire dynamic equilibrium path and fairly general functions, however.

The difference in the effects of different policy instruments suggests the possibility of mixing them optimally to reduce the cost of collecting revenue for some committed government spending. In particular, all the taxes and the subsidy on investment reduce the cost of a child (directly or indirectly), while government debt increases it, suggesting that government debt should be essential in any optimal combination of fiscal instruments.

4.3 The socially optimal policy

We now adopt a typical assumption in the related literature that the amount of government transfers is a fixed fraction of output \( gF(K_t, L_t) \) where \( g > 0 \) is a constant. The task here is to find an optimal combination of proportional taxes, investment subsidies and government debt that behaves like a lump-sum tax to finance the required spending. In doing so, we need to cancel out the wedges created by government policies in all equilibrium conditions included in Definition 1.

We have seen that canceling out the wedges in the intertemporal optimal condition requires the subsidy rate and capital income tax rate to be equal and constant over time. Also it requires the consumption tax rate to be constant over time. With endogenous fertility in the model, the subsidies and consumption taxes at constant rates and the wage
income tax reduce the cost of a child and increase fertility. But government debt has the opposite effect on fertility. Thus, the wedges of taxes, investment subsidies and government debt can be fully canceled out in the optimal condition governing fertility, for example, by setting $B_{t+1} = \tau_{wt}w_i v + s_k K_{t+1}$ with income taxes and subsidies and $B_{t+1} = \tau_c[vF_i(t) + K_{t+1}]$ with a constant consumption tax rate.

Knowing the consumer decisions as functions of government policy, the government chooses its debt and the rates of taxes and subsidies such that it can achieve the same allocation as that in the social optimum, subject to the government budget constraint (19) and its consolidated budget constraint (25). We show optimal government policy in two cases: optimal income taxation along with government debt; optimal consumption taxation along with government debt. The reason for the separation is that the former requires an additional restriction that children must not be a net financial burden. Let us use superscript $SP$ to refer to a socially optimal allocation that is taken as the target for policy making.

**Proposition 7.** For $t = 0, 1, \ldots$, if $v[1 + F_k^{SP}(t) - \delta]F_i^{SP}(t-1)/F_i^{SP}(t) < 1$ and if $g$ is sufficiently small such that $\tau_{wt} < 1$ and $\tau_{kt} < 1$, the socially optimal income taxation and government debt should be $R_{kt+1} = 1 + F_k^{SP}(t+1) - \delta$; $s_k = s_{kt+1} = \tau_{kt+1} = \tau_k < 1$; $B_{t+1} = \tau_{wt} F_i^{SP}(t) v + s_k K^{SP}_{t+1}$; and for $t = 1, 2, \ldots$, $\tau_{wt} = [1 + F_k^{SP}(t) - \delta]v[F_i^{SP}(t-1)/F_i^{SP}(t)]\tau_{wt-1} + gY_i^{SP}/F_i^{SP}(t) < 1$ such that the government budget is balanced over all periods in (25). As $t \to \infty$, $\tau_{wt}$ and $B_{t+1}$ will converge towards

$$\tau_{w\infty} = \frac{gY_i^{SP}}{F_i^{SP}(\infty)[1 - v(1 + F_k^{SP}(\infty) - \delta)]},$$

$$B_\infty = \tau_{w\infty} F_i^{SP}(\infty) v + s_k K^{SP}_\infty.$$  \(50\)

The empirically plausible restriction $v(1 + F_k(t) - \delta)F_i(t-1)/F_i(t) < 1$ for wage income taxation applies to the examples given earlier. Setting $s_k = s_{kt+1}$ at all times fully removes the wedges of capital income taxation and investment subsidization for the optimal
intertemporal allocation as mentioned earlier. Though this result has been noted in the literature on optimal taxation with exogenous fertility, it is extended to a model here with endogenous fertility whereby $s_{kt} = s_k = \tau_{kt+1}$ has a positive (negative) effect on fertility (investment) as shown in Proposition 3. Wage income taxes also have a positive (negative) effect on fertility (investment) in Proposition 4. Since government debt has opposite effects compared to those of income taxes as shown in Proposition 6, the optimal income tax has to go hand in hand with government debt such that their effects on fertility are fully canceled out. The use of wage income taxes relaxes the revenue limit in Abel (2007) and Zhang et al. (2008). More tax revenue can be collected when consumption taxation is used in our model.

We now provide optimal consumption taxation along with government debt.

**Proposition 8.** For $t = 0, 1, \ldots$, the optimal consumption taxation and government debt should be $R_{bt+1} = 1 + F_{k}^{sp}(t+1) - \delta$, $\tau_{ct} = \tau_{ct+1}, B_{t+1} = \tau_{c}[F_{i}^{sp}(t)v + K_{t+1}^{sp}]$ such that the government budget is balanced over all period in (25).

With a consumption tax, intertemporal optimality requires a constant consumption tax rate at all times, $\tau_{ct} = \tau_{ct+1}$, which generates a positive (negative) effect on fertility (investment) as shown in Proposition 5. Again, combining it with government debt is essential to remove all wedges from the optimal conditions.

In a nutshell, since all taxes have a negative effect on the cost of a child and since government debt has a positive effect on the cost of a child, the socially optimal government policy must combine taxes and government debt together for a committed lump-sum transfer. The optimal combination of them is such that their effects on fertility are fully canceled out in addition to the cancelation of tax wedges in the intertemporal optimal condition.
5. Conclusion

In this paper we have extended the existing model of neoclassical growth with endogenous fertility based on a Millian welfare function to incorporate a child cost of forgone wage income and partial capital depreciation. Since the tradeoff between investment per child and the number of children causes nonconvexity in the feasible set, the standard assumptions and conditions for social optimum with exogenous fertility are no longer applicable in our model. We have argued for the conditions for a social optimum by making an exact assumption for concave programming to deal with the nonconvexity in the feasible set. The optimal allocation can be obtained regardless of whether children’s cost is greater or smaller than their discounted earnings unlike the allocation in models where parents value the total welfare of children. The evolution of capital accumulation features an increasing tendency towards the steady state. The steady state is stable (at least locally) if higher capital intensity does not increase the marginal utility cost of a child at the steady state. In the development process with rising capital intensity, fertility tends to decline at low capital intensity (with low but fast rising wages) and may eventually start to rise near the steady state (with high but sluggish wages). The results help to explain the fertility declines and the possible eventual reversal observed in the development process.

We have explored the positive and normative implications of this model for taxation and government debt. On the positive side, we have found positive (negative) substitution effects of capital income taxes and labor income taxes on fertility (investment), a negative (positive) substitution effect of a constant consumption tax rate on investment (fertility), and a positive (negative) substitution effect of government debt on investment (fertility). These effects are shown to be valid in each period (rather than just in the steady state) with an infinite horizon. The policy implication of these results is that a country with a much higher ratio of government debt to tax revenue is likely to have a lower fertility rate and a higher saving rate. This helps to explain why Italy and Japan with much higher
government debt relative to tax revenue have much lower fertility rates without necessarily lower private saving rates than in the rest of the G-7 countries.

On the normative side, one new contribution here is the finding that government debt should be set at a level to fully offset the negative effects of taxes and investment subsidies on the cost of a child in order to achieve the social optimum. As in the literature on intertemporal capital income taxation, the subsidy rate on new investment in capital and the tax rate on net capital income should be equal to each other and remain constant over time, except the tax rate on capital income in the initial period. The wage income tax is useful in this model only when children’s costs are smaller than their discounted future earnings (a plausible situation in modern economies). Another contribution here is the derivation of the entire dynamic path of the optimal government policy comprising government debt, income tax rates and investment subsidy rates from the initial period to the long-run steady state, satisfying the overall balance of government budget for all times. The optimal wage income tax rate is increasing and convergent, while the equal rate of the capital income tax and the investment subsidy remains constant over time. The long-run optimal wage income tax rate is unique and proportional to the level of government spending, while the tax rate on capital income net of investment can generate positive net tax revenue to consolidate the government budget constraint for all times. We have also shown that it is socially optimal to tax consumption spending at a constant rate over time along with government debt such that their wedges are fully canceled out. Such an optimal consumption tax is free from the restriction on the wage income tax.

Our results of normative analysis of taxation may explain the observed positive levels of taxes on capital and labor incomes and on consumption spending, investment subsidies, and government debt. By contrast, in models where leisure is considered, wage income and consumption spending should have opposite tax rates for equal taxation of consumption and leisure in order to achieve the social optimum as shown in the literature. The opposite
taxes on consumption and wage income lack empirical support. In this sense, our focus on a tradeoff between childrearing and labor in time allocation seems to yield more realistic implications for taxation.
Appendix

A: Proof of Proposition 1. Given the definition \( W(K_t, K_{t+1}, N_t) = U(F(K_t, 1 - vN_t) + (1 - \delta)K_t - N_tK_{t+1}, N_t) \), we obtain the partial derivatives \( W_1(t) = U_c(t)[1 + F_k(t) - \delta] \), \( W_2(t) = -U_c(t)N_t \) and \( W_3(t) = -U_c(t)(vF(t) + K_{t+1}) + U_n(t) \). The planner’s problem is to solve \( \max \sum_{t=0}^{\infty} \beta^t W(K_t, K_{t+1}, N_t) \) by choosing the sequence \( (K_{t+1}, N_t)_{t=0}^{\infty} \). The Inada condition and the essential role of production inputs \( F(0, \cdot) = F(\cdot, 0) = 0 \) allow us to concentrate on an interior solution. Any interior solution to the social planner’s problem, \( \{K_{t+1}^*, N_t^*\}_{t=0}^{\infty} \), must also solve \( \max_{(K_{t+1}, N_t)}[W(K_t^*, K_{t+1}^*, N_t^*) + \beta W(K_{t+1}^*, K_{t+2}^*, N_{t+1}^*)] \), leading to necessary optimal conditions (for \( t = 0, 1, \ldots \)):

\[
W_2(K_t^*, K_{t+1}^*, N_t^*) + \beta W_1(K_{t+1}^*, K_{t+2}^*, N_{t+1}^*) = 0, \tag{52}
\]

\[
W_3(K_t^*, K_{t+1}^*, N_t^*) = 0. \tag{53}
\]

We now want to show that the social planner’s choice is at least as good as any other feasible choice \( (K_{t+1}, N_t)_{t=0}^{\infty} \) with \( K_{t+1} \geq 0 \) and \( N_t \geq 0 \):

\[
\lim_{T \to \infty} \Delta_T \equiv \lim_{T \to \infty} \sum_{t=0}^{T} \beta^t [W(K_t^*, K_{t+1}^*, N_t^*) - W(K_t, K_{t+1}, N_t)] \geq 0.
\]

The concavity of \( W(\cdot, \cdot, \cdot) \) in Assumption 1 leads to

\[
\Delta_T \geq \sum_{t=0}^{T} \beta^t \left[ W_1(K_t^*, K_{t+1}^*, N_t^*)(K_t^* - K_t) + W_2(K_{t+1}^*, K_{t+2}^*, N_{t+1}^*)(K_{t+1}^* - K_{t+1}) + W_3(K_t^*, K_{t+1}^*, N_t^*)(N_t^* - N_t) \right]
\]

\[
= \sum_{t=0}^{T-1} \beta^t \left[ W_2(K_t^*, K_{t+1}^*, N_t^*) + \beta W_1(K_{t+1}^*, K_{t+2}^*, N_{t+1}^*) \right] (K_t^* - K_{t+1}) + \beta W_1(K_0^*, K_1^*, N_1^*)(K_0^* - K_0) + \beta W_2(K_T^*, K_{T+1}^*, N_T^*)(K_{T+1}^* - K_{T+1}).
\]

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Because $K_0 = K_0$, the term $W_1(K_0^*, K_1^*, N_0^*) (K_0^* - K_0)$ in period 0 disappears from the above expression. Also, all terms containing $W_3(K_t^*, K_{t+1}^*, N_t^*) (N_t^* - N_t)$ disappear because $W_3(K_t^*, K_{t+1}^*, N_t^*) = 0$ in (53). Similarly, all terms containing $(K_{t+1}^* - K_{t+1})$ for $t < T$ drop out using (52). Finally, $eta^T W_2(K_T^*, K_{T+1}^*, N_T^*) (K_T^* - K_{T+1}) = -eta^{T+1} W_1(K_{T+1}^*, K_{T+2}^*, N_{T+1}^*) (K_{T+1}^* - K_{T+1}) = -eta^{T+1} W_1(K_{T+1}^*, K_{T+2}^*, N_{T+1}^*) K_{T+1} + eta^{T+1} W_1(K_{T+1}^*, K_{T+2}^*, N_{T+1}^*) K_{T+1}$

$\geq -\beta^{T+1} W_1(K_{T+1}^*, K_{T+2}^*, N_{T+1}^*) K_{T+1}$ since $W_1(T + 1) = U_c(T + 1) [1 + F_k(T + 1) - \delta] > 0$ and since $K_{T+1} \geq 0$. Under the transversality condition in (7), we thus end up with

$$\lim_{T \to \infty} \Delta_T \geq -\lim_{T \to \infty} \beta^{T+1} U_c(T + 1) [1 + F_k(T + 1) - \delta] K_{T+1}^* = 0.$$ 

The transversality condition essentially sets an upper bound on the concave function $\beta^T W(t)$ (Assumption 1) as time goes to infinity, leading to a summable discounted utility stream and therefore a bounded function $V(K_t)$ in the functional equation (4). In addition, the functional equation satisfies the usual monotonicity and displays discounting with a uniform metric. By Blackwell’s sufficient condition, our functional equation is a contraction with modulus $\beta$ that leads to a fixed point. A complete treatment in this regard is outside the scope of this paper. ■

B: Proof of Lemma 1. From (10), $\det(A) = \{U_{cc}(t)U_{nn}(t)N_t^2 + 2U_c(t)U_{cc}(t)N_t[vF_1(t) + K_{t+1}] - U_2(t) + U_{nn}(t) \beta V_{kk}(t+1) + U_{cc}(t) \beta V_{kk}(t+1) [vF_1(t) + K_{t+1}]^2 + U_c(t)U_{cc}(t)F_{ll}(t)v^2 N_t^2 + U_c(t)F_{ll}(t) v^2 \beta V_{kk}(t+1)\}$.

Under Assumption 1, the Hessian matrix of $W(K_t, K_{t+1}, N_t)$ must be negative definite:

$$W_{11}(t) = U_{cc}(t) [1 - \delta + F_k(t)]^2 + U_c(t) F_{kk}(t) < 0,$$

$$W_{11}(t) W_{22}(t) - W_{12}^2 = \{U_{cc}(t) [1 - \delta + F_k(t)]^2 + U_c(t) F_{kk}(t)\} U_{cc}(t) N_t^2 -$$

$$U_{cc}(t) N_t^2 [1 - \delta + F_k(t)]^2 = U_c(t) F_{kk}(t) U_{cc}(t) N_t^2 > 0,$$

$$W_{11}(t) W_{22}(t) W_{33}(t) + W_{12}(t) W_{13}(t) W_{23} + W_{12}(t) W_{13}(t) W_{23}(t) - W_{11}(t) [W_{23}(t)]^2$$
\[-W_{22}(t)[W_{13}(t)]^2 - W_{33}(t)[W_{12}(t)]^2\]

\[= U_c(t)\{[U_{cc}(t)U_{nn}(t) - U_{cn}(t)]F_{kk}(t)N_t^2 + U_c(t)U_{cc}(t)v^2N_t^2[F_{kk}(t)F_{ll}(t) - F_{kl}^2(t)]\]

\[-2U_c(t)U_{cc}(t)F_{kl}(t)vN_t[1 - \delta + F_k(t)] - U_c(t)U_{cc}(t)[1 - \delta + F_k(t)]^2 -\]

\[2U_c(t)U_{cn}(t)F_{kk}(t)N_t + 2U_c(t)U_{cc}(t)F_{kk}(t)N_t[vF_l(t) + K_{t+1}] - U_c^2(t)F_{kk}(t)\}\n
< 0.

(The elements of the Hessian matrix can be derived from the first-order derivatives $W_i$ for $i = 1, 2, 3$ given in Appendix A.)

Here, the signs of the first and second principal minors are guaranteed by the strictly increasing and strictly concave features of the production and utility functions, $F(\cdot, \cdot)$ and $U(\cdot, \cdot)$. The sign of the last one is negative if the absolute values of the second-order derivatives $U_{nn}$ and $U_{cc}$ are large enough relative to the values of the cross derivative $U_{cn}$ and to the values of first-order derivative $U_c$, noting the first-order condition $U_n(t) = U_c(t)[vF_l(t) + K_{t+1}]$. In essence, the second-order condition requires sufficiently inelastic intertemporal substitution concerning consumption across generations and sufficiently inelastic intratemporal substitution between parental consumption and fertility (or sufficiently strong relative risk aversion). The sign of the third principal minor can also be negative if the taste for fertility (included in $U_{nn}$) is strong enough relative to the taste for consumption (included in $U_c$ or $U_{cc}$). This last case is seen in the example with logarithmic utility over consumption and over fertility (unitary elasticity of substitution) in the social planner’s problem.

For the case with additively separable utility between consumption and fertility $U_{cn} = 0$, a simpler and useful implication of Assumption 1 is that $W_{22}(t)W_{33}(t) - [W_{23}(t)]^2 > 0$, leading to

\[U_{cc}(t)U_{nn}(t)N_t^2 + U_c(t)U_{cc}(t)N_t^2v^2F_{ll}(t) - U_c^2(t) + 2U_c(t)U_{cc}(t)N_t[vF_l(t) + K_{t+1}] = \]
\[
-U_{cc}(t)N_t \left[ \frac{-U_{nn}(t)N_t}{U_n(t)} - 2 \right] + U_c^2(t) \left[ \left( \frac{-U_{cc}(t)C_t}{U_c(t)} \right) \left( \frac{N_t^2 v^2(-F_{lt}(t))}{C_t} \right) \right] - 1 > 0
\]

where the first-order condition \( U_n(t) = U_c(t)[v F_l(t) + K_{t+1}] \) is used again. This sign restriction requires sufficiently large values of \( -U_{nn}N_t/U_n(t) \) and \( -U_{cc}(t)C_t/U_c(t) \) (sufficiently small values for the elasticities of intertemporal and intratemporal substitution or a strong enough taste for fertility relative to consumption). This is necessary but not fully sufficient for the first-order conditions to determine the optimal solution. Given the sign restriction under Assumption 1 here, \(|A| > 0\) because it equals \( W_{22}(t)W_{33}(t) - [W_{23}(t)]^2 \) plus the remaining terms containing \( U_{nn}(t)V_{kk}(t+1) > 0, U_{cc}(t)V_{kk}(t+1) > 0 \) or \( F_{lt}(t)V_{kk}(t+1) > 0 \).

\[ \blacksquare \]

**C: Proof of Proposition 2.** Define \( G(K_t) \equiv 1 + F_k(K_t, 1 - v N_t(K_t)) - \delta - \beta^{-1} N_t(K_t) \), which is the first factor in (15). For convergence towards the steady state, we need to find conditions for \( G'(K_t) = F_{kk}(t) - [F_{kl}(t)v + \beta^{-1}]dN_t/dK_t < 0 \). Substituting (12) into \( G'(K_t) \) gives:

\[
G'(K_t)|A| = |A|F_{kk}(t) - [F_{kl}(t)v + \beta^{-1}]\{N_tU_c(t)U_{cc}(t)[N_tvF_{kl}(t) + 1 + F_k(t) - \delta] + \beta V_{kk}(K_{t+1})[U_{cc}(t)(v F_l(t) + K_{t+1})(1 + F_k(t) - \delta) + U_c(t)vF_{kl}(t)]\}
\]

\[
= U_{cc}(t)U_{nn}(t)N_t^2 F_{kk}(t) + 2U_c(t)U_{cc}(t)N_t(v F_l(t) + K_{t+1})F_{kk}(t) - U_c^2(t)F_{kk}(t) + U_{nn}(t)\beta V_{kk}(t+1)F_{kk}(t) + U_{cc}(t)\beta V_{kk}(t+1)(v F_l(t) + K_{t+1})^2
\]

\[
F_{kk}(t) + U_c(t)U_{cc}(t)F_{lt}(t)v^2 N_t^2 F_{kk}(t) + U_c(t)F_{lt}(t)F_{kk}(t)v^2 \beta V_{kk}(t+1)
\]

\[
- [F_{kl}(t)v + \beta^{-1}]U_c(t)U_{cc}(t)N_t^2 vF_{kl}(t) - [F_{kl}(t)v + \beta^{-1}]\beta V_{kk}(t+1)
\]

\[
[U_{cc}(t)(v F_l(t) + K_{t+1})(1 + F_k(t) - \delta) + U_c(t)vF_{kl}(t)] - [F_{kl}(t)v + \beta^{-1}]
\]

\[
N_tU_c(t)U_{cc}(t)(1 + F_k(t) - \delta).
\]
In order to use the determinant of the Hessian matrix of \( W(K_t, K_{t+1}, N_t) \) in Appendix B (denoted by \(|H(W)|\)), we add and subtract two terms: \( U_c(t)U_{cc}(t)F_{kl}(t)vN_t(1 + F_k(t) - \delta) \) and \( U_c(t)U_{cc}(t)(1 + F_k(t) - \delta)^2 \). Also, we group all terms containing \( V_{kk}(t+1) \). We then rewrite \( G'(K_t)|A| \) as

\[
G'(K_t)|A| = |H(W)| + U_c(t)U_{cc}(t)[1 + F_k(t) - \delta + vN_tF_{kl}(t)][1 + F_k(t) - \delta - N_t\beta^{-1}] \\
+ V_{kk}(t+1)[\beta U_{mn}(t)F_{kk} + U_{cc}(t)\beta(vF_l(t) + K_{t+1})^2F_{kk}(t) + U_c(t)F_{kl}(t)F_{kk}(t)] \\
v^2\beta - [F_{kl}(t)v + \beta^{-1}]\beta[U_{cc}(t)(vF_l(t) + K_{t+1})(1 + F_k(t) - \delta) + U_c(t)vF_{kl}(t)] \}
\]

Assumption 1 implies \(|H(W)| < 0 \) and \(|A| > 0 \). At the steady state, \( G(K_t) = 1 + F_k(t) - \delta - N_t\beta^{-1} = 0 \). If the coefficient on \( V_{kk}(t+1) \) is nonnegative, then \( G'(K_t) < 0 \) and therefore the system is locally convergent around the steady state. This coefficient is nonnegative at the steady state under Assumption 2.

**D: Proof of Proposition 7.** The policy rules \( s_{kt} = s_{kt+1} = \tau_{kt+1} < 1 \) and \( B_{t+1} = \tau_{wt}F_l(t)v + s_kK_{t+1} \) make the consumer first-order conditions (22) and (23) be the same as (5) and (6) for the social optimum. Also, the consumer budget constraint and the government budget constraint together yield the feasibility condition (1). Moreover, for \( s_k < 1 \), the transversality conditions in (24) in the competitive equilibrium is essentially the same as that in (7) for the social optimum. Therefore, they lead to the same allocation sequence \( \{C^*_t, N^*_t, K^*_t\}^{\infty}_{t=0} \) as that for the social optimum if the optimal rates of \( s_{kt} = \tau_{kt+1} < 1 \) and \( \tau_{wt} < 1 \) for all \( t \geq 0 \) satisfy the overall balance of the government budget in (25).

Taking \( \{C^*_t, N^*_t, K^*_t\}^{\infty}_{t=0} \) and the initial state \( (B_0, K_0) \) as given, we now proceed to determine the optimal levels of government policy \( \{R_{bt}, \tau_{wt}, \tau_{kt}, s_{kt}, B_{t+1}\}^{\infty}_{t=0} \) that satisfy both the feasibility and government budget constraints in all periods (hence it satisfies the consumer budget constraint). Equations (21) and (22) lead to the no-arbitrage condition \( R_{bt+1} = R_{kt+1}/(1 - s_{kt}) = 1 + r_{t+1} - \delta = 1 + F^*_k(t + 1) - \delta \), together with \( r_t = F^*_k(t) \) in all periods.
Substituting the optimal rules of government policy \( R_{bt+1} = R_{kt+1}/(1-s_{kt}) = 1+r_{t+1}-\delta = 1+F^\text{SP}_k(t+1) - \delta; s_{kt} = s_{kt+1} = \tau_{kt+1}, T_t = gY^\text{SP}_t; B_t+1 = \tau_{wt}F^\text{SP}_t(t)v + s_kK^\text{SP}_{t+1} \) for \( t = 0, 1, \ldots \) and \( B_t = \tau_{wt-1}F^\text{SP}_t(t-1)v + s_kK^\text{SP}_t \) for \( t = 1, 2, \ldots \) into the government budget constraint in (19) yields \( \tau_{wt} = [1+F^\text{SP}_k(t) - \delta][F^\text{SP}_t(t-1)/F^\text{SP}_t(t)]\tau_{wt-1} + gY^\text{SP}_t/F^\text{SP}_t(t) \) for \( t \geq 1 \). This process is convergent towards \( \tau_{w\infty} \) in (50) under the condition \( v[1+F^\text{SP}_k(t) - \delta]F^\text{SP}_t(t-1)/F^\text{SP}_t(t) < 1 \). If \( g \) is sufficiently small, then \( \tau_{wt} < 1 \) for all \( t \geq 0 \), which can be seen in the following steps. Define the upper bounds \( 1 > a > [1+F^\text{SP}_k(t) - \delta][v[F^\text{SP}_t(t-1)/F^\text{SP}_t(t)]\tau_{wt-1} + gY^\text{SP}_t/F^\text{SP}_t(t) > 0 \) and \( b \geq gY^\text{SP}_t/F^\text{SP}_t(t) > 0 \) for all \( t \geq 0 \). Substituting the upper bounds into \( \tau_{wt} = [1+F^\text{SP}_k(t) - \delta][v[F^\text{SP}_t(t-1)/F^\text{SP}_t(t)]\tau_{wt-1} + gY^\text{SP}_t/F^\text{SP}_t(t) \), we can then obtain the upper bound on the wage income tax rate:

\[
\tau_{wt} \leq a^t\tau_{w0} + b(1-a^t)/(1-a) \quad \forall t \geq 0.
\]

When \( g \) (hence \( b \)) is small enough and when we choose \( \tau_{w0} < 1 \), we have \( \tau_{wt} < 1 \) for \( t \geq 1 \), under the restriction \( 0 < a < 1 \).

These conditions allow us to set any \( \tau_{w0} \in (0, 1) \) and update it into \( \tau_{wt} \) for \( t = 1, 2, 3, \ldots \), knowing the social optimum \( \{C^\text{SP}_t, N^\text{SP}_t, K^\text{SP}_{t+1}, Y^\text{SP}_t\}_{t=0}^{\infty} \). The time-invariant \( s_k \) for \( t \geq 0 \) and \( \tau_k \) for \( t \geq 1 \) with \( s_k = \tau_k \) can be chosen in the range \( (0, 1) \) such that \( B_{t+1} = \tau_{wt}F^\text{SP}_t(t)v + s_kK^\text{SP}_{t+1} \) for \( t = 0, 1, 2, \ldots, \infty \). In all periods, the government sets \( R_{bt} = 1+F^\text{SP}_k(t) - \delta \) so as to eliminate any arbitrage opportunities. In this way, we can pin down the path of \( \{\tau_{w,t}, s_{kt}, \tau_{kt}, R_{bt}, B_{t+1}\}_{t=0}^{\infty} \) at all times. As \( \tau_{wt} \) approaches \( \tau_{w\infty} \), \( B_t \) approaches a finite level of \( B_{\infty} \) in (51) as well.

Now, we show that the optimal government policy satisfies the overall government budget balance in all periods in (25). Because \( B_{\infty} \) is finite and because \( R_{bt} = 1+F^\text{SP}_k(t) - \delta \), it suffices to show \( \lim_{t \to \infty} \Pi^t_j(N_{j-1}/R_{bj}) \to 0 \). According to the transversality condition in (24) and the optimal condition (21), \( \lim_{t \to \infty} \beta tU_c(t)R_{bt}B_t = \lim_{t \to \infty} U_c(0)\Pi^t_j(N_{j-1}/R_{bj})R_{bt}B_t = 0 \), implying \( \lim_{t \to \infty} \Pi^t_j(N_{j-1}/R_{bj})R_{bt}B_t = 0 \) and \( \lim_{t \to \infty} \Pi^t_j(N_{j-1}/R_{bj}) = 0 \). With this result, the optimal government policy meets the overall government budget balance in (25).
when using the government budget constraint in (19) for successive substitutions:

\[ R_{t0}B_0 + \sum_{j=0}^{\infty} \Pi_j^t \left( \frac{N_{j-1}}{R_{bj}} \right) [T_i + s_{kt}(N_tK_{t+1} - K_t)] = \lim_{t \to \infty} \Pi_j^t \left( \frac{N_{j-1}}{R_{bj}} \right) N_tB_{t+1} + \]

\[ \sum_{j=0}^{\infty} \Pi_j^t \left( \frac{N_{j-1}}{R_{bj}} \right) [\tau_{wt}w_t(1 - vN_t) + \tau_{kt}(r_t - \delta)K_t] = \]

\[ \sum_{j=0}^{\infty} \Pi_j^t \left( \frac{N_{j-1}}{R_{bj}} \right) [\tau_{wt}w_t(1 - vN_t) + \tau_{kt}(r_t - \delta)K_t]. \]  

(54)

In essence, the transversality condition rules out zero tax/subsidy rates as a possible equilibrium solution. This is because inserting \( \tau_{kt} = \tau_{wt} = s_{kt} = 0 \) for all \( t \geq 0 \) into the government budget constraint in (19) leads to \( B_{t+1} = (R_{t0}/N_t)B_t + T_i/N_t \) that implies an infinitely large amount of government debt in the long run whereby \( R_{bt}/N_t > 1 \) holds in general under \( \lim_{t \to \infty} \Pi_j^t(N_{j-1}/R_{bj}) = 0 \).

The transversality condition also implies a positive amount of net tax revenue from taxing capital income and subsidizing investment at equal rates. This can be seen from signing \( \tau_{kt}(r_t - \delta)K_t - s_{kt}(N_tK_{t+1} - K_t) \). For \( \tau_{kt} = s_{kt} > 0 \), this can be rewritten as \( \tau_{kt}K_t[1 + r_t - \delta - N_tK_{t+1}/K_t] \). In the long run, the ratio \( K_{t+1}/K_t \) converges to 1 and the transversality condition leads to \( 1 + r_t - \delta = 1 + F_{k}^{sp}(t) - \delta > N_t^{sp} \) as shown earlier.

Therefore, when \( k_t \) converges to a steady state \( k^s \), \( \tau_{kt}(r_t - \delta)K_t - s_{kt}(N_tK_{t+1} - K_t) \) becomes \( \tau_{k}s[F_{k}^{sp}(t) - \delta - N_t^{sp} + 1] > 0 \) as \( t \to \infty \) under \( \tau_{kt} = s_{kt} = \tau_k > 0 \). The exact level of \( \tau_k = s_k \) is chosen such that the government budget constraint is consolidated to satisfy (54).

From \( \tau_{wt} = [1 + F_{k}^{sp}(t) - \delta]v[F_{k}^{sp}(t - 1)/F_{k}^{sp}(t)]\tau_{wt-1} + gY_{t}^{sp}/F_{k}^{sp}(t) < 1 \) for \( t \geq 1 \), we can obtain the following:

\[ \tau_{wt} = \frac{1}{F_{k}^{sp}(t)} \left\{ \Pi_{j=1}^{t} (1 + F_{k}^{sp}(j) - \delta) v^{t}F_{k}^{sp}(0)\tau_{wt0} + \sum_{i=1}^{t} \Pi_{j=1}^{i} (1 + F_{k}^{sp}(t + 2 - j) - \delta) v^{i-1}gY_{t}^{sp}_{t-i+1} \right\}, \quad \forall t \geq 1. \]
Combining this with the overall balance of the government budget for all periods in (25), it is clear that the government can simply choose \((\tau_{w0}, \tau_k)\) with \(s_k = \tau_k\) for the determination of the sequence \(\{\tau_{wt}, \tau_{kt}, s_{kt}\}_{t=0}^{\infty}\) to finance government transfers in all periods and the repayment of initial government debt in (25).

**E: Proof of Proposition 8.** The proof is similar to that given above. Consolidating the government budget over all periods with a constant \(\tau_c\) and \(B_{t+1} = \tau_c[F_t(t)v + K_{t+1}]\) gives

\[
\tau_c \sum_{t=0}^{\infty} \Pi_{j=0}^{t} \frac{N_{j-1}^{SP}}{R_{bj}} C_{jt}^{SP} = R_{b0} B_0 + \sum_{t=0}^{\infty} \Pi_{j=0}^{t} \frac{N_{j-1}^{SP}}{R_{bj}} qY_{jt}^{SP}
\]

where \(R_{bj} = 1 + F_{k}^{SP}(t) - \delta\). This determines a socially optimal \(\tau_c\).
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Table 1. Taxes, government debt, household saving and fertility in G-7 countries

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<tr>
<th>Country</th>
<th>taxes/GDP in 2005(%)&lt;sup&gt;a&lt;/sup&gt;</th>
<th>public debt/GDP in 2009 (%)&lt;sup&gt;b&lt;/sup&gt;</th>
<th>household saving 1990-2009(%)&lt;sup&gt;c&lt;/sup&gt;</th>
<th>fertility rates in 2009&lt;sup&gt;d&lt;/sup&gt;</th>
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Notes.  
<sup>b</sup> Source: IMF, World Economic Outlook, 2009.  
<sup>c</sup> Household savings as percent of disposable household income averaged for 1990-2009, Source: OECD Economic Outlook database.  
<sup>d</sup> Source: CIA World Factbook 2010.