Price Manipulation, Dynamic Informed Trading and Uniqueness of Equilibrium in a Sequential Trade Model*

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Abstract. This paper extends the Glosten and Milgrom (1985) model of asset pricing with asymmetric information into a dynamic setting and presents a model of market price manipulation. The paper shows that there is a unique equilibrium and characterizes the equilibrium. It is shown that the next-period value function of the informed trader, who knows the terminal value of the asset, is strictly monotone in terms of the market maker’s prior belief. Moreover, it is shown that bid/ask price is also a monotonically increasing of the market maker’s prior belief. The dynamic model of the informed trader is well-known to be intractable. This paper tackles this technically challenging problem and establishes the unique existence of equilibrium. Finally the paper provides a necessary and sufficient condition for manipulation to arise in equilibrium.

Key Words: Market microstructure; Glosten-Milgrom; Dynamic trading; Price formation; Sequential trade; Asymmetric information; Bid-ask spreads.

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1 Introduction

This paper develops a model of dynamic informed trading from a canonical framework in the market microstructure literature and characterizes an equilibrium. In asymmetric information models of financial markets, trading behavior imperfectly reveals the private information held by traders. Informed traders who trade dynamically thus have an incentive not only to trade less aggressively but also to manipulate the market by trading in the wrong direction, undertaking short-term losses to confuse the market and then recouping the losses in the future. Dynamic trading and price manipulation by an informed trader have been a challenging issue in the literature of market microstructure.

Despite the importance of dynamic trading strategies by informed traders in the literature, characteristics of price dynamics and information transmission have not yet been adequately studied because there is no closed-form solution for equilibrium in the dynamic Glosten-Milgrom framework and it is not yet known if equilibrium is unique either in the Kyle model (see Boulatov et al. (2005) for a further development) or the dynamic Glosten-Milgrom model in which strategic informed traders can trade repeatedly. In this paper we present a model of dynamic informed trading and show that there exists a unique equilibrium. In addition, we characterize the equilibrium bid and ask prices and specify a necessary and sufficient condition for manipulation to arise in equilibrium.

There are two standard reference frameworks in the literature. The first is called the “continuous auction framework” first developed by Kyle (1985). The second is the “sequential trade framework” proposed by Glosten and Milgrom (1985). A large amount of research has been done involving the application of these two frameworks. Both frameworks are sufficiently simple and well behaved that they easily lend themselves to analysis of policy issues and empirical testing (see Madhavan (2000) and Biais et al. (2005) for extensive surveys of the literature).

One of the simplifying assumptions in Glosten and Milgrom (1985) is that traders can trade only once. In the original Glosten-Milgrom model manipulation does not occur because there is no chance to re-trade and as a result traders maximize their one-period payoff. In the Kyle model the informed trader’s strategy is monotonic in the sense that she buys the asset when it is undervalued given her information and vice versa; dynamic price manipulation is ruled out by assumption.

This paper follows the strand of the sequential trade framework developed by Glosten and Milgrom. This paper considers markets where a risky asset is traded for finitely many periods between competitive market makers, two types of strategic informed traders and liquidity traders. In the beginning of the game, nature chooses the liquidation value of a risky asset to be high or low and tells the informed trader who trades dynamically. In each period there is a random determination of whether the informed trader or a liquidity trader trades. The market maker posts bid and ask prices for the next period, after which the trader buys or sells one unit. The termination value is revealed at the end of the game and the payoff for the informed trader is the sum of the termination value times net-holding of the asset and revenue from buying and selling the asset. Within the model described above we consider an equilibrium such that (a) the informed trader’s strategy is optimal beginning at any history; (b) market
makers make zero profit in each period under their common Bayesian belief conditional on the history and chosen trade; (c) liquidity traders trade for their exogenous liquidity needs.

The paper uses the Markov property of equilibrium to prove the uniqueness of equilibrium. In each period time and the market maker’s prior belief are the state variables. We truncate $T$-period serial problem into the problem of two-period decision making. Consider the last period. In order for manipulation to arise in equilibrium, there must be at least one more period to re-trade. Therefore in the last period manipulation would not arise in equilibrium. So we know what is the relationship between the informed trader’s payoff and the market maker’s belief due to Bayes’ rule. Given this relationship, the informed trader chooses the probability of each trade, buy or sell, in the second last period. Then we will obtain the relationship between the second last period payoff, which is the sum of second last and last period payoffs, and the market maker’s prior belief in the second last period. This is repeated recursively to the first period. Since there is no closed-form solution to the informed trader’s optimization problem, we are unable to obtain the closed-form value functions. However, we will show that if the value functions satisfy convexity and monotonicity, then there is a unique solution to the informed trader’s optimization problem. In other words, trade today will affect the prior belief tomorrow and we will prove the uniqueness of equilibrium by using backward induction. In the end we will prove that the value functions in each period satisfy convexity and monotonicity. This will establish the uniqueness of equilibrium.


The theoretical literature starts with manipulation by uninformed traders. Allen and Gale (1992) provide a model of strategic trading in which some equilibria involve manipulation. Furthermore, Allen and Gorton (1992) consider a model of pure trade-based uninformed manipulation in which an asymmetry in buys and sells by liquidity traders creates the possibility of manipulation. The first paper to consider manipulation by an informed trader within the discrete-time Glosten-Milgrom framework is Chakraborty and Yilmaz (2004). They show that when the market faces uncertainty about the existence of informed traders and there are a large number of trading periods long-lived informed traders will manipulate in every equilibrium. On the other hand, Back and Baruch (2004) study the equivalence
of the Glosten-Milgrom model and the Kyle model in a continuous-time setting, and show that the equilibrium in the Glosten-Milgrom model is approximately the same as that in the Kyle model when the trade size is small and uninformed trades arrive frequently. They conclude that the continuous-time Kyle model is more tractable than the Glosten-Milgrom model, although most markets are organized as in the sequential trade models. More recently there has been an interest in the informed trader’s dynamic strategy. Among others, Brunnermeier and Pedersen (2005) consider dynamic strategic behavior of large traders and show that “overshooting” occurs in equilibrium. Back and Baruch (2007) analyze different market systems by allowing the informed traders to trade continuously within the Glosten-Milgrom framework.

The paper is organized as follows. The next section introduces the model. Section 3 proves the uniqueness of equilibrium if the next-period value functions are strictly convex and monotone in terms of the market maker’s prior belief. Section 4 characterizes the equilibrium bid and ask prices. Section 5 proves the two property of the value functions: strict convexity and monotonicity and then the uniqueness of equilibrium. The last section concludes.

2 The model

Trade occurs for finitely many periods, denoted by \( t = 1, 2, \cdots, T \). Each interval of time accommodates one trade. There is a risky asset and a numeraire in terms of which the asset price is quoted. The terminal value of the risky asset, denoted by \( \tilde{v} \), is a random variable which can take the value 0 or 1. The risk-free interest rate is assumed to be zero.

There are three classes of risk-neutral market participants: competitive market makers, an informed trader and a liquidity trader. Trade arises from both the informed trader, who knows the terminal value of the asset, and uninformed traders. The type of the trader arriving in period \( t \) is determined by a random variable \( \tilde{\tau}_t \), which takes values from the set \( \{i, l\} \). The letters \( i \) and \( l \) respectively denote the informed type and the liquidity type. The random variables \( \{\tilde{\tau}_t : t = 1, \ldots, T\} \) satisfy \( \Pr(\tilde{\tau}_t = i) = \mu \).

There are two kinds of orders available to traders: sell or buy. Let \( A = \{S, B\} \) where \( S \) denotes a sell order and \( B \) denotes a buy order. Let \( h_t \) denote the order that the market maker receives in period \( t \), i.e. \( h_t \) is the realized order in period \( t \).

If the trader’s type in period \( t \) is \( l \), then the demand in that period is determined by the random variable \( \tilde{Q}_t \) which takes values from \( A \). The random variables \( \{\tilde{Q}_t : t = 1, \ldots, T\} \) are i.i.d. and satisfy \( \Pr(\tilde{Q}_t = B) = \gamma > 0 \). For any given period \( t \), the random variables \( \tilde{\tau}_t, \tilde{Q}_t, \tilde{v} \) are mutually independent and \( \tilde{\tau}_t, \tilde{Q}_t \) are i.i.d. across the periods \( 1, \ldots, T \).

The private information of the informed trader is determined by a random variable \( \tilde{\theta} \in \Theta = \{H, L\} \). When \( \theta = L \), the informed knows that the value of the asset is 0. We call this type of trader “low-type” and denote him by \( L \). When \( \theta = H \), the informed trader knows that the value of the asset is 1. We call this type of trader “high-type” and denote him by \( H \). Only one type of trader is actually
chosen by nature to trade for any given play of the game.

Knowledge of the game structure and of the parameters of the joint distribution of the state variables is common to all market participants. In each period market makers post bid and ask prices equal to the expected value of the asset conditional on the observed history of trades. The trader trades at those prices. Trading happens for finitely many periods after which all private information is revealed. The timing structure of the trading game is as follows:

1. In period 0 nature chooses the realization of the risky asset payoff. The informed trader observes $\theta$.

2. In successive periods, indexed by $t = 1, \ldots, T$, having observed the realized trades in periods $1, \ldots, t - 1$, the competitive market maker posts bid and ask prices. Nature chooses an informed trader of type $\theta$ with probability $\mu$ and a liquidity trader with probability $1 - \mu$. The trader learns market makers’ price quotes.

3. If the trader is informed he takes the profit-maximizing quote. If the trader is a liquidity trader he trades according to his liquidity needs.

4. In the end of period $T$, payoffs realized.

Next we describe the details with regard to market makers’ pricing strategy and the informed trader’s trading strategy. When the trader chooses his order and the market maker posts the bid and ask prices in period $t$, each knows the entire trading history until and including period $t - 1$. A period $t$ history $h_t := (h_1, \ldots, h_t)$ is a sequence of realized orders for periods until and including $t$. Let $H_t := A \times \cdots \times A$ $t$ times. We assume that: $h_0 = \emptyset$.

In the model first we consider a Markov “subgame-perfect” equilibrium and then we will show that there is not other equilibrium which does not have the Markov property. In other words we focus on the equilibrium in which market makers’ prior belief is the state variable in each period $t$. In this kind of equilibrium, we can present the next-period value of the game as a function of the prior belief in the next-period. Given that the informed trader’s optimal strategy maximizes the current-value with respect to the zero-profit prices that market makers quote. In the current period $t$, there are three possible events; the informed trader is chosen to trade, a liquidity trader is chosen to trade and buy, and a liquidity trader is chosen to trade and sell.

For each type of informed trader a trading strategy specifies a probability distribution over trades in period $t$ with respect to the bid and ask prices $p_t = (\alpha_t, \beta_t)$ posted in period $t$. The high-type informed trader buys the security with probability $\sigma^t_{H}$ and sells with probability $1 - \sigma^t_{H}$, and the low-type buys and sells with probabilities $\sigma^t_{L}$ and $1 - \sigma^t_{L}$ respectively. Since there are only trades, buy or sell, available to traders, choosing a probability of buy automatically determines a probability of sell. We call $\sigma^t = (\sigma^t_{H}, \sigma^t_{L})$ a $t$-period strategy profile.
To determine bid and ask prices to be posted in period $t$ the market maker updates his prior conditional on the arrival of an order of the relevant type. Suppose that the market makers’ prior belief at period $t$ is given by $b \in (0, 1)$. This belief, $b$, is resulted from some history $h^{t-1}$. Since we focus on the equilibrium with Markov property, we will not explicitly state $h^t$. The market maker’s prior belief after history $h^t$ on the event: $\tilde{\theta} = H$, which we denote by $\delta(h^t)$, is updated through Bayes’ rule and thus formalized as:

$$\delta(h^{t-1}, B) \equiv \Pr(\tilde{\theta} = H|h^{t-1}, B)$$

$$= \frac{b \times [\mu \sigma_H^t + (1 - \mu) \Pr(Q_t = B)]}{\mu [b \times \sigma_H^t + (1 - b) \times \sigma_L^t] + (1 - \mu) \Pr(Q_t = B)},$$

and

$$\delta(h^{t-1}, S) \equiv \frac{b \times [\mu (1 - \sigma_H^t) + (1 - \mu) \Pr(Q_t = S)]}{\mu [b \times (1 - \sigma_H^t) + (1 - b) \times (1 - \sigma_L^t)] + (1 - \mu) \Pr(Q_t = S)}.$$  

We assume that: $\delta(h^0) = \delta_0 = \Pr(\tilde{\theta} = H)$ for some $\delta_0 \in (0, 1)$. The market makers post bid and ask prices according to the zero-profit condition. Since the value of the asset is either 0 or 1, ask price in period $t$ is equal to $\delta(h^{t-1}, B)$ and bid price in period $t$ is equal to $\delta(h^{t-1}, S)$.

Now, we will define the informed trader’s optimal strategy recursively in the sense that given the continuation value of the game in the next-period the informed trader’s optimal strategy maximizes the current value of the game and this is true in all the periods. Suppose that the next-period $t + 1$ value functions $V^{t+1}_L$ and $V^{t+1}_H$ are given as a function of the market makers’ prior belief $\delta(h^t)$. Remember that as stated before we focus on an equilibrium with the Markov property in the sense that the market makers’ prior belief $b = \delta(h^{t-1})$ is the state variable at period $t$. Thus time and the market makers’ prior belief are the only state variables. Therefore the period-$t$ value of the game for each type is expressed as: for $b = \delta(h^{t-1})$, and in response to prices $p_t = (\alpha_t, \beta_t)$

$$V^t_H(b) = \max_{\sigma_H \in [0, 1]} \left[\mu \sigma_H(1 - \alpha_t + V^{t+1}_H(\alpha_t)) + (1 - \mu) \times \left[\gamma V^{t+1}_H(\alpha_t) + (1 - \gamma) V^{t+1}_H(\beta_t)\right]\right],$$

and

$$V^t_L(b) = \max_{\sigma_L \in [0, 1]} \left[\mu \sigma_L(-\alpha_t + V^{t+1}_L(\alpha_t)) + (1 - \mu) \times \left[\gamma V^{t+1}_L(\alpha_t) + (1 - \gamma) V^{t+1}_L(\beta_t)\right]\right].$$

Notice that after buy or sell order at period $t$, the market makers’ posterior belief becomes $\alpha_t$ or $\beta_t$ and thus the next-period value of the game for each type becomes a function of those variables.

**Definition 1.** *The high-type informed trader’s strategy $\{\sigma^t_H : t = 1, \cdots, T\}$ is optimal in response to prices $\{p_t : t = 1, \cdots, T\}$ if it prescribes a probability $\{\sigma^t_H : t = 1, \cdots, T\}$ such that for every $t$, $\sigma^t_H$...*
solves (3) in response to $p_t$. The low-type informed trader’s strategy \{\sigma_{t}^{*} : t = 1, \cdots , T\} is optimal in response to prices \{p_t : t = 1, \cdots , T\} if it prescribes a probability \{\sigma_{t}^{*} : t = 1, \cdots , T\} such that for every $t$, \sigma_{t}^{*}$ solves (4) in response to $p_t$.

Next we define an equilibrium for our economy:

**Definition 2.** An equilibrium consists of a pair of bid and ask prices \{\sigma_{t}^{*} = (\alpha_{t}^{*}, \beta_{t}^{*})\}_{t \in \{1, \cdots , T\}} and an informed trader’s strategies \{\sigma_{t}^{*}\}_{t = 1, \cdots , T}$ such that for all $t \in \{1, \cdots , T\}$ and for every $b = \delta(h_{t}^{-1})$ with $h_{t}^{-1} \in \mathcal{H}_{t}^{-1}$,

\begin{enumerate}[(P1)]  
  \item the pair of bid and ask prices $p_{t}^{*}$ satisfies the zero-profit condition with respect to the market maker’s posterior belief;
  \item the informed trader’s strategy profile \{\sigma_{t}^{*}\}_{t = 1, \cdots , T}$ is optimal given the pair of bid and ask prices $p_{t}^{*}$;
  \item the pair of bid and ask prices $p_{t}^{*} = (\alpha_{t}^{*}, \beta_{t}^{*})$ satisfies Bayes’ rule.
\end{enumerate}

Now, we define a manipulative strategy. We say that a strategy is manipulative if it involves the informed trader undertaking a trade in any period that yields a strictly negative short-term profit. If this occurs in equilibrium it means that manipulation enables the informed trader to recoup the short-term losses.

**Definition 3.** In response to a pair of bid and ask prices $p_{t}$ for some $t \in \{1, \cdots , T\}$, a strategy profile \{\sigma\}_{t = 1, \cdots , T}$ is called manipulative for the high type in period $t$ if $\sigma_{t}^{H} < 1$; or for the low type if $\sigma_{t}^{L} > 0$. Moreover we say that \{\sigma\}_{t = 1, \cdots , T}$ is manipulative for both types in period $t$ if both conditions hold; or for only one type if only one of the two conditions holds.

This is the same definition used by Chakraborty and Yilmaz (2004). Back and Baruch (2004) used the term “bluffing” instead. We call the situation where the informed trader chooses a totally mixed strategy “price manipulation.” It’s worth mentioning that in Huberman and Stanzl (2004) a price manipulation is defined as a round-trip trade. In this paper price manipulation occurs as a round-trip trade in equilibrium but not by definition. This is because if the informed trader trades against his short term profit incentive he incurs a loss which must be recouped, consequently price manipulation takes the form of a round-trip trade in equilibrium.

3 Existence of Equilibrium

3.1 Arc-convex Functions

Consider a monotone function $f$ defined $[0, 1]$. We define the following term:
Definition 4. • Suppose that \( f(x) \) is increasing in \( x \). Then the monotonically increasing function \( f \) is defined to be arc-convex.

• Suppose that \( f(x) \) is decreasing in \( x \). Then the monotonically increasing function \( f \) is defined to be arc-concave.

Definition 5. • Suppose that \( f(x) \) is increasing in \( x \). Then the monotonically increasing function \( f \) is defined to be arc-convex.

• Suppose that \( f(x) \) is decreasing in \( x \). Then the monotonically increasing function \( f \) is defined to be arc-concave.

Proposition 1. Suppose that a function \( f \) is monotonically increasing and arc-convex on \([0, 1]\). Then the following holds:

1. a strictly convex function is arc-convex;
2. for every \( h > 0 \), \( \frac{f(x + h) - f(x)}{h} > \frac{f(x)}{x} \), especially \( f'(x) > \frac{f(x)}{x} \).

Proofs of Proposition 1 - 1. Suppose that \( f \) is strictly convex. Then we have:

\[
\left( \frac{f(x)}{x} \right)' = \frac{xf'(x) - f(x)}{x^2}.
\] (5)

Strict convexity of \( f \) implies \( f'(x) > \frac{f(x)}{x} \), which completes our proof.

Proofs of Proposition 1 - 2. On the contrary, suppose that \( \frac{f(x + h) - f(x)}{h} \leq \frac{f(x)}{x} \). Then we have

\[
\frac{f(x + h)}{x + h} = \frac{f(x + h) - f(x) + f(x)}{h + x} \leq \frac{f(x)}{h + x} = \frac{(h + x) \cdot f(x)}{x \cdot (h + x)} = \frac{f(x)}{x},
\] (6)

which contradicts with the first statement that \( \frac{f(x)}{x} \) is increasing in \( x \). Therefore we conclude:

\[
\frac{f(x + h) - f(x)}{h} > \frac{f(x)}{x}.
\] (7)

By taking the limit of \( h \) to zero, we can obtain \( f'(x) > \frac{f(x)}{x} \) and this completes our proof.

In this section we will prove the uniqueness of equilibrium if the next-period value function is strictly monotonic and convex in market makers’ prior belief \( b \). Take an arbitrary time period \( t \). Since our plan is to use the Markov property of equilibrium, we will focus on the time period \( t \) given the property of the next-period value function in terms of market makers’ prior in the next-period \( t + 1 \). So in this section for the simplicity of notation we will eliminate \( t \) or \( t + 1 \) from superscript or subscript in each variable unless specified. Fix a history \( h^t \) arbitrarily and let \( b = \delta(h^t) \in (0, 1) \) be market makers’ prior belief after history \( h_t \). Let \( W_H = V_H^t \) and \( W_L = V_L^t \) represent the current value of the game for
both traders. Let $V_H = V_H^{t+1}$ and $V_L = V_L^{t+1}$ represent the continuation value of the remainder of the game for both traders. In other words, $W_H$ and $W_L$ are the value functions in the current period and $V_H$ and $V_L$ are the value functions in the next period. Suppose that the next-period value function $V_H$, and $V_L$ satisfy the following two conditions:

**M** $V_H$ is strictly decreasing in $[0, 1]$ and $V_L$ is strictly increasing in $[0, 1]$;

**C** $V_H$ and $V_L$ are continuous in $[0, 1]$;

and there exist $m_H$ and $m_L$ with $m_H > m_L$ such that:

**A** $V_H$ is arc-convex in $[m_H, 1]$ and $V_L$ are arc-convex in $[0, m_L]$;

**S** $V_H$ is strictly convex in $[0, m_H]$ and $V_L$ are strictly convex in $[m_L, 1]$.

### 3.2 Existence of Equilibrium

For arbitrary belief $b$, define the best-response correspondence $BR^b = (BR^b_M, BR^b_H, BR^b_L)$ as follows:

$$BR^b_M(\sigma_L, \sigma_H) = p = \{(\alpha, \beta) : (\alpha, \beta) \text{ satisfies Bayes' rule (1) & (2)}\};$$

$$BR^b_H(p) = \{\sigma_H : \sigma_H \text{ solves (3)}\};$$

$$BR^b_L(p) = \{\sigma_L : \sigma_L \text{ solves (4)}\}.$$

**Theorem 1.** The equilibrium exists if conditions (M) and (C) hold.

**Proof of Theorem 1.** Then, the equilibrium $(p^*, \sigma^*_H, \sigma^*_L)$ is defined as a fixed-point of $BR^b$. When $t = T$, we must have: $\sigma^*_H = 1$ and $\sigma^*_L = 0$. Then fix $t \in \{1, \ldots, T - 1\}$ arbitrarily. Notice that $[0, 1]$ is a non-empty compact convex set. To apply Kakutani’s fixed point theorem, we have to prove that $BR^b$ is upper semi-continuous, convex and non-empty mapping.

**Non-Emptiness:** Fix $p$ arbitrarily and then $V_H(b)$ becomes a continuous function of $\sigma_H$. For all $t < \infty$, $V_H^{t+1}$ is well-defined and $\sigma_H$ is defined in a compact set $[0, 1]$. Therefore there is $\sigma_H$ which solves (3). Thus $BR^b_H(p)$ is non-empty. Similarly we can say that $BR^b_L(p)$ is non-empty. Fix $\sigma$ arbitrarily and then $BR^b_M$ is non-empty and thus $BR^b$ is non-empty. □

**Upper Semi-continuity:** It is clear that $BR^b_M$ is continuous in $\sigma$. It remains to show that $BR^b_H$ and $BR^b_L$ are upper semi-continuous. On the contrary, suppose that $BR^b_H$ is not upper semi-continuous. Then there are two sequences $(p^k)$ converging to $p$ and $(\sigma^k_H)$ converging to $\sigma_H$ such that for every $k \sigma^k_H \in BR^b_H(p^k)$ but $\sigma_H \notin BR^b_H(p)$. Then there must be a different $\hat{\sigma}_H \in BR^b_H(p)$ such that: for some $\epsilon > 0$,

$$\mu(\sigma_H(1 - \alpha + V_H(\alpha)) + (1 - \sigma_H)(\beta - 1 + V_H(\beta))) + (1 - \mu) \times [\gamma V_H(\alpha) + (1 - \gamma)V_H(\beta)] > \mu(\sigma_H(1 - \alpha + V_H(\alpha)) + (1 - \sigma_H)(\beta - 1 + V_H(\beta))) + (1 - \mu) \times [\gamma V_H(\alpha) + (1 - \gamma)V_H(\beta)] + 5\epsilon.$$
Since $V_H$ is continuous and $p^k \to p$ and $\sigma^k_H \to \sigma_H$, for each $k$ there exists a collection of strictly positive numbers $(\epsilon^i_1, \ldots, \epsilon^i_4)$ such that: for all $k' \geq k$,

$$|\alpha - \alpha^{k'}| < \epsilon^i_1; |\beta - \beta^{k'}| < \epsilon^i_2; |V_H(\alpha) - V_H(\alpha^{k'})| < \epsilon^i_3; |V_H(\beta) - V_H(\beta^{k'})| < \epsilon^i_4. \quad (8)$$

Take $k_1$ sufficiently large so that we have: $\epsilon > \max(\epsilon^i_1, \ldots, \epsilon^i_4)$. Notice that: for all $k \geq k_1$,

$$\begin{align*}
\mu(\tilde{\sigma}_H(1 - \alpha + V_H(\alpha)) + (1 - \tilde{\sigma}_H)(\beta - 1 + V_H(\beta))) + (1 - \mu) \times [\gamma V_H(\alpha) + (1 - \gamma)V_H(\beta)]
- \mu(\tilde{\sigma}_H(1 - \alpha^k + V_H(\alpha^k)) + (1 - \tilde{\sigma}_H)(\beta^k - 1 + V_H(\beta^k))) - (1 - \mu) \times [\gamma V_H(\alpha^k) + (1 - \gamma)V_H(\beta^k)]
\leq \mu(\tilde{\sigma}_H(\epsilon^1_{3k} + \epsilon^2_{3k}) + (1 - \tilde{\sigma}_H)(\epsilon^2_{3k} + \epsilon^4_{3k})) + (1 - \mu) \times (\gamma \epsilon^3_{3k} + (1 - \gamma)\epsilon^4_{3k})
\leq 2\mu\epsilon + (1 - \mu)\epsilon < 2\epsilon.
\end{align*}$$

Therefore, for all $k \geq k_1$, we obtain:

$$\begin{align*}
\mu(\tilde{\sigma}_H(1 - \alpha + V_H(\alpha)) + (1 - \tilde{\sigma}_H)(\beta - 1 + V_H(\beta))) + (1 - \mu) \times [\gamma V_H(\alpha) + (1 - \gamma)V_H(\beta)]
\geq \mu(\tilde{\sigma}_H(1 - \alpha^k + V_H(\alpha^k)) + (1 - \tilde{\sigma}_H)(\beta^k - 1 + V_H(\beta^k))) + (1 - \mu) \times [\gamma V_H(\alpha^k) + (1 - \gamma)V_H(\beta^k)] + \epsilon.
\end{align*}$$

Then take $k_2$ sufficiently large so that we have:

$$\epsilon > \left[\mu(2 - \alpha^{k_2} + V_H(\alpha^{k_2}) - \beta^{k_2} - V_H(\beta^{k_2}))(\sigma_{H_B}^{k_2} - \sigma_H)\right], \quad (9)$$

and then we have: for all $k \geq k_2$,

$$\begin{align*}
\mu(\tilde{\sigma}_H(1 - \alpha + V_H(\alpha)) + (1 - \tilde{\sigma}_H)(\beta - 1 + V_H(\beta))) + (1 - \mu) \times [\gamma V_H(\alpha) + (1 - \gamma)V_H(\beta)]
\geq \mu(\tilde{\sigma}_H(1 - \alpha^k + V_H(\alpha^k)) + (1 - \tilde{\sigma}_H)(\beta^k - 1 + V_H(\beta^k))) + (1 - \mu) \times [\gamma V_H(\alpha^k) + (1 - \gamma)V_H(\beta^k)] + \epsilon.
\end{align*}$$

This contradicts with our assumption: $\sigma^k_H \in BR^k_H(p^k)$ for all $k$. We can prove the desired result for $BR^k_{H'}$ in a similar fashion. Finally we conclude that $BR^k$ is upper semi-continuous. □

**Convexity:** Given $\sigma$, $BR^k_M(\sigma_L, \sigma_H)$ consists of a unique point $p$ which satisfies Bayes’ rule. Therefore $BR^k_M$ is convex. Now suppose that: $\sigma_H, \tilde{\sigma}_H \in BR^k_H(p)$. Then for any $r \in (0, 1)$ we have $r\sigma_H + (1 - r)\tilde{\sigma}_H \in BR^k_H(p)$. This implies that:

$$\mu(\tilde{\sigma}_H(1 - \alpha + V_H(\alpha)) + (1 - \tilde{\sigma}_H)(\beta - 1 + V_H(\beta))) + (1 - \mu) \times [\gamma V_H(\alpha) + (1 - \gamma)V_H(\beta)]
= \mu(\tilde{\sigma}_H(1 - \alpha + V_H(\alpha)) + (1 - \tilde{\sigma}_H)(\beta - 1 + V_H(\beta))) + (1 - \mu) \times [\gamma V_H(\alpha) + (1 - \gamma)V_H(\beta)].$$

Then we obtain:

$$(\tilde{\sigma}_H - \sigma_H)(1 - \alpha + V_H(\alpha)) + ((1 - \tilde{\sigma}_H) - (1 - \sigma_H))(\beta - 1 + V_H(\beta)) = 0.$$
\[ r\sigma_H + (1 - r)\hat{\sigma}_H = \sigma_H \in BR_H^b(p) \] by our assumption. If \( 1 - \alpha + V_H(\alpha) = \beta - 1 + V_H(\beta) \), then for any \( r \in (0, 1) \) and \( \bar{\sigma}_H = r\sigma_H + (1 - r)\hat{\sigma}_H \) we have:

\[
\bar{\sigma}_H(1 - \alpha + V_H(\alpha)) + (1 - \bar{\sigma}_H)(\beta - 1 + V_H(\beta)) = \sigma_H(1 - \alpha + V_H(\alpha)) + (1 - \sigma_H)(\beta - 1 + V_H(\beta)).
\]

Then we conclude that: \( \bar{\sigma}_H \in BR_H^b(p) \). We can prove the desired result for \( BR_L^b(p) \) in a similar fashion. Finally we conclude that \( BR^b \) is convex. \( \square \)

Finally, by Kakutani’s fixed point theorem, we conclude that: the equilibrium exists in period \( t \). Since in period \( t = T \) the equilibrium exists and \( V_L^T \) and \( V_H^T \) are both well-defined, by using mathematical induction we conclude that the equilibrium exists in all periods. \( \square \)

After Theorem 1 the equilibrium correspondence and the graph in each period \( t \) are well-defined and given by:

\[
\mathcal{E}(b) = \{\sigma : \sigma \text{ is an equilibrium strategy profile for belief } b\};
\]

\[
\mathcal{G}(b) = \{(b, \sigma) : \sigma \in \mathcal{E}(b)\}.
\]

**Proposition 2.** Suppose that conditions (M) and (C) hold. The equilibrium correspondence \( \mathcal{E} \) is upper semi-continuous and the graph \( \mathcal{G} \) is closed on \([0, 1]\).

**Proof of Proposition 2.** Now on the contrary, suppose that \( \mathcal{E} \) is not upper semi-continuous. Then there is a sequence \( \{b^k\} \) which converges to \( b \) and \( \sigma^k \) which converges to \( \sigma \) with \( \sigma^k \in \mathcal{E}(b^k) \) for every \( k \) but \( \sigma \notin \mathcal{E}(b) \). We respectively denote the sequences of prices which satisfy Bayes’ rule with \( \sigma^k \) and \( b^k \) by \( p^k \) and \( \sigma \) and \( b \) by \( p \). Then there must be \( \hat{\sigma} \) with \( \hat{\sigma} \in \mathcal{E}(b) \) and there exists \( \epsilon > 0 \) such that:

\[
\mu(\bar{\sigma}_H(1 - \alpha + V_H(\alpha)) + (1 - \bar{\sigma}_H)(\beta - 1 + V_H(\beta))) + (1 - \mu) \times [\gamma V_H(\alpha) + (1 - \gamma)V_H(\beta)]
\]

\[
> \mu(\sigma_H(1 - \alpha + V_H(\alpha)) + (1 - \sigma_H)(\beta - 1 + V_H(\beta))) + (1 - \mu) \times [\gamma V_H(\alpha) + (1 - \gamma)V_H(\beta)] + 5\epsilon.
\]

Since \( p \) is continuous in \( \sigma \) and \( b \), we must have: \( p^k \to p \). Take \( k_1 \) sufficiently large so that we have:

\[
\epsilon > \max(\varepsilon_1^{k_1}, \cdots, \varepsilon_4^{k_1}) \quad (\varepsilon_1^{k_1}, \text{ to } \varepsilon_4^{k_1} \text{ are defined in (8)}) \text{.}
\]

Then by continuity of the value function (which condition (C) implies) and prices, a similar argument with (3.2) gives us:

\[
\mu(\bar{\sigma}_H(1 - \alpha^k + V_H(\alpha^k)) + (1 - \bar{\sigma}_H)(\beta^k - 1 + V_H(\beta^k))) + (1 - \mu) \times [\gamma V_H(\alpha^k) + (1 - \gamma)V_H(\beta^k)]
\]

\[
> \mu(\sigma_H(1 - \alpha^k + V_H(\alpha^k)) + (1 - \sigma_H)(\beta^k - 1 + V_H(\beta^k))) + (1 - \mu) \times [\gamma V_H(\alpha^k) + (1 - \gamma)V_H(\beta^k)] + 3\epsilon.
\]

Since \( \sigma^k \to \sigma \), we can take \( k_2 \) which satisfies (9). Then for all \( k \geq k_2 \), we obtain:

\[
\mu(\bar{\sigma}_H(1 - \alpha^k + V_H(\alpha^k)) + (1 - \bar{\sigma}_H)(\beta^k - 1 + V_H(\beta^k))) + (1 - \mu) \times [\gamma V_H(\alpha^k) + (1 - \gamma)V_H(\beta^k)]
\]

\[
\mu(\sigma_H(1 - \alpha^k + V_H(\alpha^k)) + (1 - \sigma_H)(\beta^k - 1 + V_H(\beta^k))) + (1 - \mu) \times [\gamma V_H(\alpha^k) + (1 - \gamma)V_H(\beta^k)] + \epsilon
\]

\[
> \mu(\bar{\sigma}_H(1 - \alpha^k + V_H(\alpha^k)) + (1 - \bar{\sigma}_H)(\beta^k - 1 + V_H(\beta^k))) + (1 - \mu) \times [\gamma V_H(\alpha^k) + (1 - \gamma)V_H(\beta^k)] + \epsilon.
\]

This contradicts the assumption that \( \sigma^k \in \mathcal{E}(b^k) \) for all \( k \). The closedness of the graph \( \mathcal{G} \) follows. \( \square \)
To simplify our proof we will make use of the symmetric setting of the model. Let: \( \tilde{b} = 1 - b \) and \( \tilde{\gamma} = 1 - \gamma \). Consider the same situation with our original economy except that now liquidity buys with probability \( \tilde{\gamma} \) and market maker’s belief is set as \( \tilde{b} \). We call this economy “mirror economy.” In what follows, \( \tilde{\cdot} \) stands for variables associated with the mirror economy.

**Proposition 3.** Suppose that conditions \((M)\) and \((C)\) hold. Fix time \( t \) and prior \( b = \delta(h^{t-1}) \).

1. Let \( \sigma \in \mathcal{E}(b) \) and \( \tilde{\sigma}_L = 1 - \sigma_H, \tilde{\sigma}_H = 1 - \sigma_L \). Then we have: \( \tilde{\sigma} \in \tilde{\mathcal{E}}(\tilde{b}) \).
2. Let \((\alpha, \beta)\) denote the equilibrium prices associated with \( \sigma \) in the original economy and \((\tilde{\alpha}, \tilde{\beta})\) be the equilibrium prices associated with \( \tilde{\sigma} \) in the mirror economy. Then, we have: \( \alpha = 1 - \tilde{\beta}, \beta = 1 - \tilde{\alpha} \).
3. For every \( t \), we have: \( V^t_H(b) = \tilde{V}^t_H(\tilde{b}) \) and \( V^t_H(b) = \tilde{V}^t_L(\tilde{b}) \).

**Proof of Proposition 3.** By definition the period-\( t \) value of the game for each type in the mirror economy is expressed as: for \( \tilde{b} = 1 - b = 1 - \delta(h^{t-1}) \), and in response to prices \( \tilde{\sigma} = (\tilde{\alpha}, \tilde{\beta}) \)

\[
\tilde{V}^t_H(\tilde{b}) = \max_{\tilde{\sigma}_H \in [0,1]} \left( \mu \tilde{\sigma}_H (1 - \tilde{\alpha} + V^{t+1}_H(\tilde{\alpha})) + \mu (1 - \tilde{\sigma}_H) (\tilde{\beta} - 1 + V^{t+1}_H(\tilde{\beta})) + (1 - \mu) \times \left( \tilde{\gamma} V^{t+1}_H(\tilde{\alpha}) + (1 - \tilde{\gamma}) V^{t+1}_H(\tilde{\beta}) \right) \right),
\]

and

\[
\tilde{V}^t_L(\tilde{b}) = \max_{\tilde{\sigma}_L \in [0,1]} \left( \mu \tilde{\sigma}_L (-\tilde{\alpha} + V^{t+1}_L(\tilde{\alpha})) + \mu (1 - \tilde{\sigma}_L) (\tilde{\beta} - 1 + V^{t+1}_L(\tilde{\beta})) + (1 - \mu) \times \left( \tilde{\gamma} V^{t+1}_L(\tilde{\alpha}) + (1 - \tilde{\gamma}) V^{t+1}_L(\tilde{\beta}) \right) \right).
\]

Also Bayes’ rule dictates:

\[
\tilde{\alpha} = \frac{\mu \tilde{\sigma}_H + (1 - \mu) \tilde{\gamma}}{(1 - \mu) \tilde{\gamma} + \mu \tilde{\sigma}_L (1 - \tilde{b}) + \mu \tilde{\sigma}_H \tilde{b} \cdot \tilde{b};
\]

and

\[
\tilde{\beta} = \frac{\mu (1 - \tilde{\sigma}_H) + (1 - \mu) (1 - \tilde{\gamma})}{(1 - \mu) (1 - \tilde{\gamma}) + \mu (1 - \tilde{\sigma}_L) (1 - \tilde{b}) + \mu (1 - \tilde{\sigma}_H) \cdot \tilde{b} \cdot \tilde{b}.}
\]

Having the description of the equilibrium in the mirror economy, now we consider the relationship of the two equilibria in the original economy and mirror economy recursively. When \( t = T \), we have: \( \tilde{\sigma}_L = (1 - \sigma_H) = 1 \) and \( 1 - \tilde{\sigma}_H = \sigma_L = 0 \) because they do not manipulate in the last period, and so \( I \) is proved. Then by Bayes’ rule, (12) and (13), we have: \( \alpha = 1 - \tilde{\beta}, \beta = 1 - \tilde{\alpha} \), which proves 2., and also since there is no more chance to trade, the equalities of those prices and the comparison of (3) and (11) or (4) and (10) give us: \( V^T_L(b) = \tilde{V}^T_H(\tilde{b}) \) and \( V^T_H(b) = \tilde{V}^T_L(\tilde{b}) \). This gives us 3. and completes the proof for this case. \( \square \)

When \( t \neq T \), suppose that \( \sigma \in \mathcal{E}(b) \) and \((\alpha, \beta)\) is the equilibrium prices associated with \( \sigma \) in the original economy. Moreover suppose that the next-period value functions satisfy the property that 3. describes. Let: \( \tilde{\sigma}_{LB} = (1 - \sigma_H), \tilde{\sigma}_{HS} = \sigma_L \). Then we have 2. because:

\[
\alpha = 1 - \tilde{\beta} \quad \text{and} \quad \beta = 1 - \tilde{\alpha}.
\]
By substituting 2. into (11) and \( \dot{V}_{L}^{t+1} \), applying 3. to \( \dot{V}_{L}^{t+1} \), we obtain:

\[
(11) = \max_{\sigma_{H} \in \Delta(A)} \left( \mu \sigma_{H} (1 - \alpha t + V_{H}^{t+1}(\alpha t)) \right)
+ \mu (1 - \sigma_{H}) (\beta t - 1 + V_{H}^{t+1}(\beta t)) + (1 - \mu) \left[ \gamma V_{H}^{t+1}(\alpha t) + (1 - \gamma) V_{H}^{t+1}(\beta t) \right] = V_{H}^{t}(b),
\]
and similarly by substituting 2. into (10) and \( \dot{V}_{H}^{t+1} \), applying 3. to \( \dot{V}_{H}^{t+1} \), we obtain:

\[
(10) = \max_{\sigma_{L} \in \Delta(A)} \left( \mu \sigma_{L} (-\alpha t + V_{L}^{t+1}(\alpha t)) + \mu (1 - \sigma_{L}) (\beta t + V_{L}^{t+1}(\beta t)) \right)
+ (1 - \mu) \left[ \gamma V_{L}^{t+1}(\alpha t) + (1 - \gamma) V_{L}^{t+1}(\beta t) \right] = V_{L}^{t}(b).
\]

This shows that the current-period value functions also satisfy 3. and it remains to show that 1. satisfies. If \( \bar{\sigma} \not\in \hat{E}(\bar{b}) \), then there must be a different strategy profile \( \bar{\sigma} \in \hat{E}(\bar{b}) \), which indicates that there is a different strategy profile \( \bar{\sigma} \in E(b) \). This is a contradiction to our assumption. □

Since the results hold for the last period \( T \), by mathematical induction we conclude that the results hold for all the periods.

3.3 Preliminary Results

Lemma 1. Let \( \sigma \in E(b) \) and \( p = (\alpha, \beta) \) be an equilibrium price associated with \( \sigma \) and \( b \). The followings hold:

1. \( \alpha > b > \beta \);
2. \( \sigma_{H} > \sigma_{L} \).

Proof of Lemma 1 - 1. On the contrary suppose that for some \( b \), \( \alpha \leq \beta \). Notice that for \( b \in (0, 1) \) \( \alpha \) or \( \beta \) cannot be either 0 nor 1 by Bayes’ rule. Then, by (M) we have:

\[
1 - \alpha + V_{H}(\alpha) > \beta - 1 + V_{H}(\beta) \quad (17)
\]
\[
-\alpha + V_{L}(\alpha) < \beta + V_{L}(\beta). \quad (18)
\]

Then, it must be the case that in equilibrium \( \sigma_{H} = 1 \) and \( \sigma_{L} = 0 \). Then, by Bayes’ rule, we must have: \( \alpha > b > \beta \), which contradicts with our assumption. □

Proof of Lemma 1 - 2. The result follows from 1 and Bayes’ rule. □

In equilibrium the high-type trader will not sell with probability one and the low-type trader will not buy with probability one. This means that an informed trader either trades on his information or assigns a positive probability to both buy and sell orders. In the latter case the informed trader is indifferent between buy and sell orders. This motivates the following lemma.
Lemma 2. Let $\sigma \in \mathcal{E}(b)$ and $p = (\alpha, \beta)$ be an equilibrium price associated with $\sigma$ and $b$. Then, the following holds:

$$W_H(b, \sigma) = \mu (1 - \alpha + V_H(\alpha)) + (1 - \mu) (\gamma V_H(\alpha) + (1 - \gamma)V_H(\beta)),$$

and

$$W_L(b, \sigma) = \mu (\beta + V_L(\beta)) + (1 - \mu) (\gamma V_L(\alpha) + (1 - \gamma)V_L(\beta)).$$

(19)  (20)

Proof of Lemma 2. Omitted.

Next, we consider the slopes of the value functions. By Lemma 1 bid-ask spread $\alpha - \beta$ is strictly positive. If the low-type manipulates we have:

$$V_L(\alpha) - V_L(\beta) = \frac{\alpha + \beta}{\alpha - \beta} = 1 + \frac{2\beta}{\alpha - \beta} > 1.$$  (21)

Similarly if the high-type manipulates we have:

$$V_H(\alpha) - V_H(\beta) = \frac{\alpha + \beta - 2}{\alpha - \beta} = -1 - \frac{2 - 2\alpha}{\alpha - \beta} < -1.$$  (22)

This means that if the low-type manipulates, then the average slope between the ask and bid prices in the value function is greater than 1. A similar argument also holds for the high-type. Thus we conclude the following.

Lemma 3. Let: $\sigma \in \mathcal{E}(b)$ and $p = (\alpha, \beta)$ be an equilibrium price associated with $\sigma$ and $b$.

L. If the low-type takes a manipulative strategy at $b$, then

$$\lim_{h \to 0} \frac{V_L(\alpha + h) - V_L(\alpha)}{h} > 1.$$

H. If the high-type takes a manipulative strategy at $b$, then

$$\lim_{h \to 0} \frac{V_H(\beta + h) - V_H(\beta)}{h} < -1.$$

Proof of Lemma 3. Consider:

$$V_L(\alpha(b)) - V_L(\beta(b)) = \alpha(b) + \beta(b).$$  (23)

From the indifference condition we can see that: $V_L(\alpha(b)) > \alpha(b)$. This implies: $V'_L(\alpha(b)) > \frac{V_L(\alpha(b))}{\alpha(b)} > 1$. We can prove the desired result for H, in a similar fashion.

Define $b_L$ and $b_H$ as follows:

$$b_L = \inf \{ b : \sigma_L > 0 \} \quad \text{and} \quad b_H = \inf \{ b : \sigma_H < 1 \}.$$

Then arc-convexity of $V$’s gives us the following result.

Corollary 1. Suppose $b_L, b_H \in (0, 1)$. Then for all $b \geq b_L$, $V'_L(b) > 1$ and for all $b \leq b_H$, $V'_H(b) < -1$. 

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3.4 Four Regimes

Let $\sigma \in \mathcal{E}(b)$ and $p = (\alpha, \beta)$ be an equilibrium price associated with $\sigma$ and $b$. Depending on the equilibrium strategy, we can classify the equilibrium into four regimes:

**Regime $L$:** $\sigma$ is manipulative for only the low-type;

**Regime $H$:** $\sigma$ is manipulative for only the high-type;

**Regime $\emptyset$:** $\sigma$ is not manipulative;

**Regime $LH$:** $\sigma$ is manipulative for both types.

We denote the set of equilibrium strategies when Regime $i$ arises at belief $b$ by $R_i(b)$ for $i \in \{L, \cdots, LH\}$ and more formally it is defined as:

\[
R_L(b) = \{\sigma \in \mathcal{E}(b) : \sigma_H = 1 \& \sigma_L > 0\};
R_H(b) = \{\sigma \in \mathcal{E}(b) : \sigma_H < 1 \& \sigma_L = 0\};
R_\emptyset(b) = \{\sigma \in \mathcal{E}(b) : \sigma_H = 1 \& \sigma_L = 0\};
R_{LH}(b) = \{\sigma \in \mathcal{E}(b) : \sigma_H < 1 \& \sigma_L > 0\}. \tag{24}
\]

We also define $I_i = \{b \in (0, 1) : \exists \sigma \in R_i(b)\}$. There is a possibility that several regimes co-exist in some beliefs. The next Lemmas prove that different regimes do not co-exist.

**Lemma 4.** *Regime LH does not co-exist with Regime $\emptyset$.*

*Proof of Lemma 4.* Let $\sigma \in R_{LH}(b)$ and suppose by way of contradiction $\hat{\sigma} \in R_\emptyset(b)$. Let $(\alpha, \beta)$ be an equilibrium price associated with $\sigma$, and $(\hat{\alpha}, \hat{\beta})$ be an equilibrium price associated with $\hat{\sigma}$, respectively. By Bayes’ rule $\hat{\alpha} > \alpha$ and $\hat{\beta} < \beta$. We have:

\[
1 - \alpha + V_H(\alpha) = \beta - 1 + V_H(\beta); \tag{25}
1 - \hat{\alpha} + V_H(\hat{\alpha}) \geq \hat{\beta} - 1 + V_H(\hat{\beta}). \tag{26}
\]

Subtracting (26) from (25) yields:

\[
\hat{\alpha} - \alpha - V_H(\hat{\alpha}) + V_H(\alpha) \leq \hat{\beta} - \beta + V_H(\beta) - V_H(\hat{\beta}). \tag{27}
\]

By Lemma 3 we know that: $V'_H(\beta) < -1$. By condition (C) and $\hat{\beta} < \beta$, the right hand side of (26) is strictly smaller than 0. However, since $V_H$ is decreasing by condition (M), the left hand side of (26) is strictly greater than 0, which makes (26) impossible to hold. \qed

**Lemma 5.** *Regime $L$ or $H$ does not co-exist with Regime $\emptyset$.***
Proof of Lemma 5. Since the argument is symmetric, we will prove the result for $R_H(b)$. Let $\sigma \in R_H(b)$ and suppose by way of contradiction $\hat{\sigma} \in R_0(b)$. Let $(\alpha, \beta)$ be an equilibrium price associated with $\sigma$, and $(\hat{\alpha}, \hat{\beta})$ be an equilibrium price associated with $\hat{\sigma}$, respectively. Since $(1-\sigma_L) = \hat{\sigma}_{LS} = 1$, we have by Bayes’ rule $\hat{\alpha} > \alpha$ and $\hat{\beta} < \beta$. We have:

$$1 - \alpha + V_H(\alpha) = \beta - 1 + V_H(\beta); 1 - \hat{\alpha} + V_H(\hat{\alpha}) \geq \hat{\beta} - 1 + V_H(\hat{\beta}).$$

(28)

Subtracting (28) from (28) yields:

$$\hat{\alpha} - \alpha - V_H(\hat{\alpha}) + V_H(\alpha) \leq \beta - \hat{\beta} + V_H(\beta) - V_H(\hat{\beta}).$$

(29)

By Lemma 3 we know that: $V'_H(\beta) < -1$. By condition (C) and $\hat{\beta} < \beta$, the right hand side of (63) is strictly smaller than 0. However, since $V_H$ is decreasing by condition (M), the left hand side of (63) is strictly greater than 0, which makes (63) impossible to hold.

Lemma 6. Regime LH does not co-exist with Regime L or H.

Proof of Lemma 6. Let $\sigma \in R_{LH}(b)$ and suppose by way of contradiction $\hat{\sigma} \in R_H(b)$. Let $(\alpha, \beta)$ be an equilibrium price associated with $\sigma$, and $(\hat{\alpha}, \hat{\beta})$ be an equilibrium price associated with $\hat{\sigma}$, respectively. We have:

$$1 - \alpha + V_H(\alpha) = \beta - 1 + V_H(\beta); 1 - \hat{\alpha} + V_H(\hat{\alpha}) = \hat{\beta} - 1 + V_H(\hat{\beta}).$$

(30)

(31)

and

$$-\alpha + V_L(\alpha) = \beta + V_L(\beta);$$

$$-\hat{\alpha} + V_L(\hat{\alpha}) \leq \hat{\beta} + V_L(\hat{\beta}).$$

(32)

(33)

Case 1: $\alpha > \hat{\alpha}$: Since $\hat{\alpha} > \hat{\beta}$, we must have:

$$\hat{\alpha} = \frac{b}{b + \bar{l}_{H}(b)(1-b)} > \beta = \frac{b}{b + \frac{(1-\bar{l})}{(1-h_H(b))}(1-b)}.$$ 

Therefore we must have:

$$\frac{\bar{l}}{h_H(b)} < \frac{(1-\bar{l})}{(1-h_H(b))}. $$

(34)

Now by Bayes’ Rule, we must have:

$$\alpha = \frac{h(b)\bar{l}}{h(b)\bar{l} + \bar{h}(b)(1-\bar{l})} > \hat{\alpha} = \frac{h_H(b)\bar{l}}{h_H(b)\bar{l} + \bar{h}(b)(1-\bar{l})}.$$ 

$$\iff \bar{l}h(b) > h_H(b)l(b).$$

(35)
\[ \bar{\alpha} > \hat{\alpha} \rightarrow \beta < \hat{\beta}. \]  

By condition (M) the LHS of (43) is strictly smaller than 0. If \( \beta < \hat{\beta} \) then by Lemma 3 we must have: \( \frac{V_H(\hat{\alpha}) - V_H(\hat{\beta})}{\hat{\beta} - \beta} < -1 \) which indicates that the RHS of (43) is strictly greater than 0. This is a contradiction. □

Case 2: \( \alpha \geq \hat{\alpha} \): Then by Proposition 3 in the mirror economy, we must have: \( 1 - \hat{\beta} \geq 1 - \hat{\beta} \). Therefore we have: \( \hat{\beta} \leq \beta \). Then by taking the contrapositive of (42), we obtain \( \hat{\beta} \leq \beta \rightarrow \alpha \leq \hat{\alpha} \). By applying this result to the mirror economy, we obtain: \( \hat{\beta} \leq \beta \rightarrow \hat{\alpha} \leq \alpha \), which indicates: \( 1 - \beta \leq 1 - \hat{\beta} \). Thus, we must have: \( \hat{\beta} \leq \beta \).

Now subtracting (33) from (32) yields:

\[ \hat{\alpha} - \alpha - V_L(\hat{\alpha}) + V_H(\alpha) \geq \beta - \hat{\beta} + V_L(\beta) - V_H(\hat{\beta}). \]  

Since \( \hat{\beta} \leq \beta \), by condition (M) we must have: the RHS of (44) ≥ 0. If \( \alpha = \hat{\alpha} \), then we must have \( \beta = \hat{\beta} \), which is impossible because as the proof in 8, if both prices are the same, then \( \sigma = \hat{\sigma} \). This is a contradiction with our assumption. Now suppose \( \alpha < \hat{\alpha} \). However by condition (C) we have: \( \frac{V_L(\hat{\alpha}) - V_L(\alpha)}{\alpha - \hat{\alpha}} > 1 \) and thus the LHS of (44) < 0. Therefore (44) is impossible. □
Lemma 7. Regime $L$ does not co-exist with Regime $H$.

Proof of Lemma 7. By way of contradiction suppose that at prior $b$ Regime $L$ and Regime $H$ co-exist.

We denote one pair of prices associated with Regime $L$ by $(\alpha_1, \beta_1)$, and the other associated with Regime $H$ by $(\alpha_2, \beta_2)$. Then by the indifference condition we must have:

\[ -\alpha_1 + V_L(\alpha_1) = \beta_1 + V_L(\beta_1); \] (45)
\[ 1 - \alpha_1 + V_H(\alpha_1) \geq \beta_1 - 1 + V_H(\beta_1); \] (46)

and also

\[ -\alpha_2 + V_L(\alpha_2) \leq \beta_2 + V_L(\beta_2); \] (47)
\[ 1 - \alpha_2 + V_H(\alpha_2) = \beta_2 - 1 + V_H(\beta_2). \] (48)

Consider Bayes’ rule:

\[ \alpha_1 = \frac{\bar{h}b}{\bar{h}b + l(b)(1-b)} \quad \& \quad \alpha_2 = \frac{h(b)b}{h(b)b + l(1-b)}; \]
\[ \beta_1 = \frac{(1-\bar{h})b}{(1-h)b + (1-l(b))(1-b)} \quad \& \quad \beta_2 = \frac{(1-h(b))b}{(1-h(b))b + (1-l)(1-b)}. \]

Case 1: $\alpha_1 > \alpha_2$: Let $\bar{h} = (1-\mu)\gamma + \mu$ and $\bar{l} = (1-\mu)\gamma$ so that we have: $h(b) \leq \bar{h}$ and $l(b) \geq \bar{l}$.

Since $\alpha_2 > \beta_2$, we must have:

\[ \alpha_2 = \frac{b}{b + \frac{\bar{l}}{h(b)}(1-b)} > \beta_2 = \frac{b}{b + \frac{(1-\bar{l})}{(1-h(b))}(1-b)}. \]

Therefore we must have:

\[ \frac{\bar{l}}{h(b)} < \frac{(1-\bar{l})}{(1-h(b))}. \] (49)

Now by Bayes’ Rule, we must have:

\[ \alpha_1 = \frac{\bar{h}b}{\bar{h}b + l(b)(1-b)} > \alpha_2 = \frac{h(b)b}{h(b)b + l(1-b)}; \]

\[ \iff \quad \bar{h}\bar{l} > h(b)l(b). \] (50)

\[ \iff \quad \bar{h}\bar{l} - h(b)l(b) > h(b)l(b) - h(b)\bar{l}. \] (51)

\[ \iff \quad (\bar{h} - h(b))\bar{l} > h(b)(l(b) - \bar{l}). \] (52)
\[ (\bar{h} - 1 + 1 - h(b))\bar{l} > h(b)(l(b) - 1 + 1 - \bar{l}). \]  
\[ \iff \]  
\[ (1 - \frac{1 - \bar{h}}{1 - h(b)})\bar{l}(1 - h(b)) > h(b)(1 - \bar{l})(1 - \frac{1 - l(b)}{1 - l}). \]  
\[ \iff \]  
\[ (1 - \frac{1 - \bar{h}}{1 - h(b)}) \frac{\bar{l}}{1 - l} > h(b) \frac{1}{1 - h(b)} (1 - \frac{1 - l(b)}{1 - l}). \]  

By (49) we must have: \( \frac{\bar{l}}{1 - l} < \frac{h(b)}{1 - h(b)} \). In addition \( \frac{1 - \bar{h}}{1 - h(b)} < 1 \) and \( \frac{1 - l(b)}{1 - l} < 1 \). Therefore in order for (55) to hold, we must have:

\[ 1 - \frac{1 - \bar{h}}{1 - h(b)} > 1 - \frac{1 - l(b)}{1 - l}. \]  

Therefore we must have: \( \frac{1 - h}{1 - h(b)} < \frac{1 - l(b)}{1 - l} \), which indicates \( \beta_1 < \beta_2 \) by Bayes’ rule. Now subtracting (48) from (46) yields:

\[ -\alpha_1 + \alpha_2 + V_H(\alpha_1) - V_H(\alpha_2) \geq \beta_1 - \beta_2 + V_H(\beta_1) - V_H(\beta_2). \]  

(57)

Since \( \beta_1 < \beta_2 \), by condition (C) we have: \( \frac{V_H(\beta_1) - V_H(\beta_2)}{\beta_1 - \beta_2} < -1 \), which indicates the RHS of (57) > 0. However \( \alpha_1 \geq \alpha_2 \) indicates the LHS of (57) \( \leq 0 \). Therefore (57) is impossible. □

Case 2: \( \alpha_2 \geq \alpha_1 \): Suppose that we have: \( \alpha_2 \geq \alpha_1 \). Then by Proposition 3 in the mirror economy, we must have: \( 1 - \beta_2 \geq 1 - \beta_1 \). Therefore we have: \( \beta_2 \leq \beta_1 \). Then by taking the contrapositive of the proved statement in the first part of this proof, which is: \( \alpha_1 > \alpha_2 \rightarrow \beta_1 < \beta_2 \), we obtain: \( \beta_2 \leq \beta_1 \rightarrow \alpha_1 \leq \alpha_2 \). By applying this result to the mirror economy, we obtain: \( \tilde{\beta}_2 \leq \tilde{\beta}_1 \rightarrow \tilde{\alpha}_1 \leq \tilde{\alpha}_2 \), which indicates: \( 1 - \beta_1 \leq 1 - \beta_2 \). Therefore we conclude: \( \beta_2 \leq \beta_1 \).

Now subtracting (45) from (47) yields:

\[ \alpha_1 - \alpha_2 + V_L(\alpha_2) - V_L(\alpha_1) \leq \beta_2 - \beta_1 + V_L(\beta_2) - V_L(\beta_1). \]  

(58)

Since \( \beta_1 \geq \beta_2 \), by condition (M) we must have: the RHS of (58) \( \leq 0 \). However \( \alpha_1 < \alpha_2 \) indicates by condition (C) we have: \( \frac{V_L(\alpha_2) - V_L(\alpha_1)}{\alpha_2 - \alpha_1} > 1 \) and thus the LHS of (58) > 0. Therefore (58) is impossible. □

**Lemma 8.** Suppose that in equilibrium, there are two different pairs of prices \((\alpha, \beta)\) associated with the equilibrium strategy \(\sigma\) and \((\hat{\alpha}, \hat{\beta})\) associated with the equilibrium strategy \(\hat{\sigma}\). If \(\alpha = \hat{\alpha}\) and \(\beta = \hat{\beta}\), then \(\sigma = \hat{\sigma}\).

**Proof of Lemma 8.** For the simplicity of notation, let \( h = (1 - \mu)\gamma + \mu\sigma_H \) and \( l = (1 - \mu)\gamma + \mu\sigma_L \). Suppose that in equilibrium, there are two different pairs of strategies, \(\sigma\) and \(\hat{\sigma}\) in \(R_{LH}\). Similarly with
Let \( \sigma \in \mathcal{E}(b) \) and \((\alpha, \beta)\) denote the equilibrium ask and bid prices. We define the following functions:

\[
A(b, \sigma_L, \sigma_H) = \frac{[(1-\mu)\gamma + \mu \sigma_H]b}{(1-\mu)(1-\gamma) + \mu \sigma_H + \mu(1-\sigma_H)\sigma_L},
\]

\[
B(b, \sigma_L, \sigma_H) = \frac{[(1-\mu)(1-\gamma) + \mu \sigma_H + \mu(1-\sigma_H)\sigma_L]b}{(1-\mu)(1-\gamma) + \mu b(1-\sigma_H) + \mu(1-b)(1-\sigma_L)}.
\]

4 Within Each Regime

Let \( \sigma \in \mathcal{E}(b) \) and \((\alpha, \beta)\) denote the equilibrium ask and bid prices. We define the following functions:
By the definition of the functions, obviously we have:

\[ A(b, \sigma^L, \sigma^H) = \alpha \quad & \quad B(b, \sigma^L, \sigma^H) = \beta. \] (66)

Define the profits of trades for each type as follows:

\[
\begin{align*}
\pi_{HB}(b, \sigma_L, \sigma_H) &= 1 - A(b, \sigma_L, \sigma_H) + V_H(A(b, \sigma_L, \sigma_H)); \\
\pi_{HS}(b, \sigma_L, \sigma_H) &= B(b, \sigma_L, \sigma_H) - 1 + V_H(b, \sigma_L, \sigma_H)); \\
\pi_{LB}(b, \sigma_L, \sigma_H) &= -A(b, \sigma_L, \sigma_H) + V_L(A(b, \sigma_L, \sigma_H)); \\
\pi_{LS}(b, \sigma_L, \sigma_H) &= B(b, \sigma_L, \sigma_H) + V_L(b, \sigma_L, \sigma_H)).
\end{align*}
\]

Define \( \sigma^H_L, \sigma^H_H \) and \( \sigma^L_H, \sigma^H_H \) as follows:

\[
\begin{align*}
\pi_{LB}(b, \sigma^L_L, 1) &= \pi_{LS}(b, \sigma^L_L, 1) \quad \text{and} \quad \pi_{HB}(b, \sigma^H_L, 1) = \pi_{HS}(b, \sigma^H_L, 1); \\
\pi_{LB}(b, 0, \sigma^H_H) &= \pi_{LS}(b, 0, \sigma^H_H) \quad \text{and} \quad \pi_{HB}(b, 0, \sigma^H_H) = \pi_{HS}(b, 0, \sigma^H_H). 
\end{align*}
\] (67)

**Lemma 10.**

- As \( \sigma_H \) increases, \( A(b, \sigma_L, \sigma_H) \) increases and \( B(b, \sigma_L, \sigma_H) \) decreases.
- As \( \sigma_L \) decreases, \( A(b, \sigma_L, \sigma_H) \) increases and \( B(b, \sigma_L, \sigma_H) \) decreases.

**Proof of Lemma 10.** Proved by Bayes’ rule. \( \square \)

**Lemma 11.**

- If there exists \( \sigma^L_H \in (0, 1) \) satisfying \( \pi_{LB}(b, 0, \sigma^L_H) = \pi_{LS}(b, 0, \sigma^L_H) \), then \( \pi_{LB}(b, 0, \sigma) \) is monotonically increasing as \( \sigma \) increases for all \( \sigma \geq \sigma^L_H \).
- If there exists \( \sigma^L_L \in (0, 1) \) satisfying \( \pi_{LB}(b, \sigma^L_L, 1) = \pi_{LS}(b, \sigma^L_L, 1) \), then \( \pi_{LB}(b, \sigma, 1) \) is monotonically decreasing as \( \sigma \) decreases for all \( \sigma \leq \sigma^L_L \).

**Proof of Lemma 11.** When \( \pi_{LB}(b, 0, \sigma^L_H) = \pi_{LS}(b, 0, \sigma^L_H) \) holds, we have:

\[
V_L(A(b, 0, \sigma^L_H)) - V_L(B(b, 0, \sigma^L_H)) = A(b, 0, \sigma^L_H) + B(b, 0, \sigma^L_H),
\] (68)

which dividing by \( A(b, 0, \sigma^L_H) - B(b, 0, \sigma^L_H) \), and arc-convexity of \( V_L \) give us \( V_L'(A(b, 0, \sigma)) > 1 \) for all \( \sigma \geq \sigma^L_H \). For \( \Delta > 0 \), we have:

\[
\begin{align*}
\pi_{LB}(b, 0, \sigma + \Delta) - \pi_{LB}(b, 0, \sigma + \Delta) &= V_L(A(b, 0, \sigma + \Delta)) - V_L(A(b, 0, \sigma)) - A(b, 0, \sigma + \Delta) + A(b, 0, \sigma) \\
&= (A(b, 0, \sigma + \Delta) - A(b, 0, \sigma)) \left( \frac{V_L(A(b, \sigma + \Delta)) - V_L(A(b, \sigma))}{A(b, \sigma + \Delta) - A(b, \sigma)} - 1 \right).
\end{align*}
\]

By Bayes’ rule, \( A(b, 0, \sigma) \) is monotonically increasing in \( \sigma \). As \( V_L'(A(b, 0, \sigma)) > 1 \) for all \( \sigma \geq \sigma^L_H \), we obtain \( \pi_{LB}(b, 0, \sigma + \Delta) > \pi_{LB}(b, 0, \sigma + \Delta) \). The second statement can be proved symmetrically to the first statement. \( \square \)
Now we consider equilibrium within each regime. In Regime $\emptyset$, there is only one equilibrium strategy in which the low-type sells and the high-type buys with probability 1. So we focus on the other regimes. We consider a profit from taking a “honest strategy” for each type. If Regime $\emptyset$ arises, then the following two conditions must be holding:

\[
\begin{align*}
\pi_{HB}(b, 0, 1) &\geq \pi_{HS}(b, 0, 1); \\
\pi_{LB}(b, 0, 1) &\leq \pi_{LS}(b, 0, 1).
\end{align*}
\] (69)

**Lemma 12.** When $\sigma_H = \sigma_L = \sigma$, the following holds:

\[
\begin{align*}
\pi_{HB}(b, \sigma, \sigma) &> \pi_{HS}(b, \sigma, \sigma); \\
\pi_{LB}(b, \sigma, \sigma) &< \pi_{LS}(b, \sigma, \sigma).
\end{align*}
\] (70)

**Proof of Lemma 12.** By Bayes’ rule, when $\sigma_H = \sigma_L = \sigma$, we obtain: for any $b$,

\[A(b, \sigma, \sigma) = B(b, \sigma, \sigma) = b.\]

By substituting the above into $\pi$’s, we obtain the desired result. □

**Lemma 13.**

- If $\pi_{LB}(b, 0, 1) \leq \pi_{LS}(b, 0, 1)$, then the low-type does not manipulate in equilibrium.
- If $\pi_{HB}(b, 0, 1) \geq \pi_{HS}(b, 0, 1)$, then the high-type does not manipulate in equilibrium.

**Proof of Lemma 13.** Take a decreasing sequence of the low-type’s strategies denoted by $\{\sigma^k\}$, which goes from 1 to 0. On the contrary, suppose that the low-type manipulates. Then there has to be $\sigma^K$ such that:

\[
\begin{align*}
\pi_{HB}(b, \sigma^K, 1) &\geq \pi_{HS}(b, \sigma^K, 1); \\
\pi_{LB}(b, \sigma^K, 1) &= \pi_{LS}(b, \sigma^K, 1).
\end{align*}
\] (71)

Then for all $K > K$ (i.e., $\sigma^K < \sigma^K$), we obtain $\pi_{LB}(b, \sigma^K, 1) > \pi_{LS}(b, \sigma^K, 1)$, which leads us to $\pi_{LB}(b, 0, 1) > \pi_{LS}(b, 0, 1)$. Contradiction. The second statement can be proved symmetrically. □

**Lemma 14.** Suppose that the followings hold:

\[
\begin{align*}
\pi_{HB}(b, 0, 1) &\leq \pi_{HS}(b, 0, 1); \\
\pi_{LB}(b, 0, 1) &\geq \pi_{LS}(b, 0, 1).
\end{align*}
\] (72)

If $\sigma_H^L \leq \sigma_L^H$ holds, the low-type manipulates and if $\sigma_L^H \geq \sigma_H^L$ holds, the high-type manipulates.

**Proof of Lemma 14.** □

**Proposition 4.** Regime LH does not arise in equilibrium.
Proof of Proposition 4. Suppose that Regime $H$ arises till $b$ and at $b$ Regime $LH$ arises. Then for any sufficiently small $\epsilon > 0$, there exists a $\sigma_{H}^{H\epsilon}$ such that:

$$
\pi_{HB}(b - \epsilon, 0, \sigma_{H}^{H\epsilon}) = \pi_{HS}(b - \epsilon, 0, \sigma_{H}^{H\epsilon});
$$
$$
\pi_{LB}(b - \epsilon, 0, \sigma_{H}^{H\epsilon}) \leq \pi_{LS}(b - \epsilon, 0, \sigma_{H}^{H\epsilon}).
$$

Then there must exist a $\sigma_{L}^{H\epsilon}$ with $\sigma_{L}^{H\epsilon} \geq \sigma_{H}^{H\epsilon}$ such that:

$$
\pi_{HB}(b - \epsilon, 0, \sigma_{L}^{H\epsilon}) \leq \pi_{HS}(b - \epsilon, 0, \sigma_{L}^{H\epsilon});
$$
$$
\pi_{LB}(b - \epsilon, 0, \sigma_{L}^{H\epsilon}) = \pi_{LS}(b - \epsilon, 0, \sigma_{L}^{H\epsilon}).
$$

As the two different regimes do not co-exist, for $\sigma_{H}^{H} > \sigma_{L}^{H}$ we obtain:

$$
\pi_{HB}(b, 0, \sigma_{H}^{H}) = \pi_{HS}(b, 0, \sigma_{H}^{H});
$$
$$
\pi_{LB}(b, 0, \sigma_{H}^{H}) > \pi_{LS}(b, 0, \sigma_{H}^{H}).
$$

As $\pi_{HB}$ and $\pi_{HS}$ are continuous in $b$ and $\sigma_{H}$, $\sigma_{H}^{H\epsilon}$ converges to $\sigma_{H}^{H}$ and similarly as $\pi_{LB}$ and $\pi_{LS}$ are continuous in $b$ and $\sigma_{H}$, $\sigma_{L}^{H\epsilon}$ converges to $\sigma_{L}^{H}$. As the equilibrium exists uniquely in Regime $H$, $\sigma_{H}^{H\epsilon}$ is continuous. Therefore, for every $\epsilon > 0$ there exists a $\delta > 0$ such that:

$$
(\pi_{LB}(b - \delta, 0, \sigma_{H}^{H\delta}) - \pi_{LS}(b - \delta, 0, \sigma_{H}^{H\delta})) - (\pi_{LB}(b, 0, \sigma_{H}^{H}) - \pi_{LS}(b, 0, \sigma_{H}^{H})) < \epsilon.
$$

However, we have:

$$
(\pi_{LB}(b - \delta, 0, \sigma_{H}^{H\delta}) - \pi_{LS}(b - \delta, 0, \sigma_{H}^{H\delta})) \leq 0,
$$

while at the limit,

$$
\pi_{LB}(b, 0, \sigma_{H}^{H}) - \pi_{LS}(b, 0, \sigma_{H}^{H}) = \epsilon' > 0.
$$

This is a contradiction.

Proposition 5. • If the following holds:

$$
\pi_{HB}(b, 0, 1) \leq \pi_{HS}(b, 0, 1) \quad \text{and} \quad \pi_{LB}(b, 0, 1) \leq \pi_{LS}(b, 0, 1),
$$

then in equilibrium Regime $H$ arises.

• If the following holds:

$$
\pi_{HB}(b, 0, 1) \geq \pi_{HS}(b, 0, 1) \quad \text{and} \quad \pi_{LB}(b, 0, 1) \geq \pi_{LS}(b, 0, 1),
$$

then in equilibrium Regime $L$ arises.
Proposition 6. The equilibrium exists uniquely when the value functions in every period satisfy the conditions (M) and (C).

Proof of Proposition 6. Proposition 4 indicates: \( I_{LH} = \emptyset \). Lemma 5 indicates that \( I_L \cap I_\emptyset = \emptyset \) and \( I_H \cap I_\emptyset = \emptyset \). Lemma 7 indicates that \( I_L \cap I_H = \emptyset \). In the end Proposition ?? completes our proof.

Proposition 7. The equilibrium strategy \( \sigma = \mathcal{E}(b) \) is continuous in \( b \) on \((0, 1)\).

Proof of Proposition 7. By Proposition 6, \( \mathcal{E} \) is a function of prior belief \( b \). By Theorem 1 the equilibrium correspondence \( \mathcal{E} \) is upper semi-continuous. Therefore we conclude that it is continuous.

Lemma 15. Bid and ask prices are continuous in \( b \).

Proof of Lemma 15. By Lemma 7 and Bayes’ rule, the result follows.

Lemma 16. Consider the boundary of each regime, which is a prior \( b \in (0, 1) \) such that for any sufficiently small \( \epsilon > 0 \), \( b - \epsilon \in I_i \) and \( b + \epsilon \in I_j \) for \( j \neq i \) and \( i, j \in \{L, H, \emptyset\} \). Let \( \sigma = \mathcal{E}(b) \) and then we must have: \( \sigma_H = 1 \) and \( \sigma_L = 0 \).

Proof of Lemma 16. By Proposition 6 and Proposition 7, the equilibrium strategy is expressed by a continuous function of prior belief denoted by \( \mathcal{E}(b) \). The result is clear when \( I_L \leftrightarrow I_\emptyset \) and \( I_H \leftrightarrow I_\emptyset \) because of the definition of Regime \( \emptyset \) and the continuity of the equilibrium strategy. Consider \( I_L \leftrightarrow I_H \). Since the argument is symmetric, we only consider \( I_L \rightarrow I_H \). Take a sequence \( \{\epsilon_k\}_k \) with \( \epsilon_k > 0 \) which converges to 0. For every \( b_k = b - \epsilon_k \), we have \( \sigma_k = R_L(b_k) \) and every \( \hat{b}_k = b + \epsilon_k \), we have \( \hat{\sigma}_k = R_H(\hat{b}_k) \). Take a limit of \( k \) to infinity. Then we have \( \sigma_k \rightarrow \sigma \) and \( \hat{\sigma}_k \rightarrow \sigma \) by the continuity of the equilibrium strategy. The only possibility is: \( \sigma_H = 1 \) and \( \sigma_L = 0 \) by the continuity of the equilibrium strategy.

Lemma 17. The high-type manipulates in the region where \( V_H \) is strictly convex. Similarly the low-type manipulates in the region where \( V_L \) is strictly convex.

Proof of Lemma 17. Notice that both \( V_H \) and \( V_L \) are strictly convex in the region of \([m_L, m_H]\). As \( V_H \) is symmetric to \( V_L \) with respect to \( \frac{1}{2} \), we have \( m_H = 1 - m_L \). Suppose that the high-type manipulates in the region of \((0, b_H)\) and the low-type manipulates in the region of \((b_L, 1)\). Then again \( b_H = 1 - b_L \). If \( b_H > m_H \), then we must have \( 1 - b_L > 1 - m_L \). Thus both types manipulate in the region of \((b_L, b_H)\). This is a contradiction.
5 Characterization of Bid and Ask Prices

To simplify the arguments, we impose one assumption of \( \gamma = \frac{1}{2} \). Notice that in this case \( V_H \) is symmetric to \( V_L \) with respect to \( \frac{1}{2} \).

We redefine ask and bid functions as \( \alpha : \Delta(\Theta) \rightarrow [0, 1] \) and \( \beta : \Delta(\Theta) \rightarrow [0, 1] \). Moreover we extend the definitions for the functions \( h(b) \) and \( l(b) \) to all regimes, that is, for \( \sigma \in \mathcal{E}(b) \), \( h(b) = (1 - \mu)\gamma + \mu \sigma_H \), \( l(b) = (1 - \mu)\gamma + \mu \sigma_L \). In addition, define: \( P(b) = h(b) \times b + l(b) \times (1 - b) \). In words \( P \) is the market maker’s expectation that buy order comes.

**Lemma 18.** Ask and bid prices, \( \beta(b) \) and \( \alpha(b) \), satisfy the following:

1. for every prior \( b \in (0, 1) \), \( 1 > \alpha(b) > 0 \) and \( 1 > \beta(b) > 0 \);
2. for every prior \( b \in (0, 1) \), bid-ask spread is non-zero: that is, \( \alpha(b) - \beta(b) \neq 0 \).

**Proof of Lemma 18.** 1. For every \( b \in (0, 1) \), \( 1 > P(b) > 0 \) by Lemma 1. In addition \( h(b)b \neq 0 \) and \( (1 - h(b))b \neq 0 \). By Bayes’ rule we conclude: \( 1 > \alpha(b) > 0 \) and \( 1 > \beta(b) > 0 \). □

2. The proof is done by Lemma 1. □

**Lemma 19.** In each regime, bid and ask prices are piecewise differentiable in every period. Moreover the value functions are piecewise differentiable in every period.

**Proof of Lemma 19.** Notice that: \( \sigma_H \in R_2(b) \) or \( \sigma_L \in R_1(b) \) solves the following equation:

\[
1 - \frac{b \times [\mu \sigma_H + (1 - \mu)\gamma]}{\mu b \times \sigma_H + (1 - \mu)\gamma} + V_H(\frac{b \times [\mu \sigma_H + (1 - \mu)\gamma]}{\mu b \times \sigma_H + (1 - \mu)\gamma}) = \frac{b \times [\mu (1 - \sigma_H) + (1 - \mu)(1 - \gamma)]}{\mu b \times [\mu + (1 - \mu)\gamma]} - 1 + V_H(\frac{b \times [\mu (1 - \sigma_H) + (1 - \mu)(1 - \gamma)]}{\mu b \times [\mu + (1 - \mu)\gamma]}); \tag{81}
\]

\[
\frac{b \times [\sigma_L + (1 - \mu)(1 - \gamma)]}{\mu b \times [\sigma_L + (1 - \mu)(1 - \gamma)]} + V_L(\frac{b \times [\sigma_L + (1 - \mu)(1 - \gamma)]}{\mu b \times [\sigma_L + (1 - \mu)(1 - \gamma)]}) = \frac{b \times (1 - \gamma)}{\mu (1 - b) \times \sigma_L + (1 - \mu)(1 - \gamma)} + V_L(\frac{b \times (1 - \gamma)}{\mu (1 - b) \times \sigma_L + (1 - \mu)(1 - \gamma)}).
\]

Notice that: \( V^T \)'s are continuously differentiable on \( [0, 1] \). Suppose that the next-period value functions are piecewise differentiable. By the Implicit Function Theorem, \( \sigma_H \) or \( \sigma_L \) are piecewise continuously differentiable in terms of \( b \). Bid and ask prices are a continuously differentiable function in terms of \( b \) or \( \sigma_H \) or \( \sigma_L \). Therefore we conclude that bid and ask prices are piecewise differentiable. By Lemma 2 the current value function is also piecewise differentiable. By mathematical induction the result holds for every period. □

Suppose that the low-type manipulates. Let: \( F_l(b) = l(b) - l'(b)b(1 - b) \). Then, we have:

\[
\alpha'(b) = \frac{\bar{h} \cdot F_l(b)}{P(b)} \quad \text{and} \quad \beta'(b) = \frac{(1 - \bar{h}) \cdot (1 - F_l(b))}{1 - P(b)}.
\]

Therefore, we have:

\[
\alpha'(b)b = \frac{\alpha(b) \cdot F_l(b)}{P(b)} \quad \text{and} \quad \beta'(b)b = \frac{\beta(b) \cdot (1 - F_l(b))}{1 - P(b)} \tag{82}.
\]
and
\[
\alpha''(b) = h(b) \times \frac{F'_I(b) P(b) - F_I(b) \times 2P'(b)}{(h(b)b + l(b)(1-b))^3};
\]
\[
\beta''(b) = [1 - h(b)] \times \frac{-F'_I(b)(1 - P(b)) + (1 - F_I(b)) \times 2P'(b)}{([1 - h(b)]b + [1 - l(b)](1-b))^3}.
\]

**Proposition 8.** Suppose that the next-period value functions satisfy (C) and (M). Ask and bid prices, \(\alpha(b)\) and \(\beta(b)\) are strictly increasing in \(b\).

**Proof of Proposition 8.** When nobody manipulates, by the Bayes rule we can show that bid and ask prices decrease as market makers’ belief \(b\) decreases. It remains to show that the result holds in Regime \(L\) and \(H\). Since the argument is symmetric, suppose that the low-type manipulates at \(b\). Then, as the low-type’s indifference condition for \(\sigma\) we have:

\[-\alpha(b) + V_L(\alpha(b)) = \beta + V_L(\beta(b)).\]  
(85)

Taking the first derivative we obtain:

\[\alpha'(b)(-1 + V'_L(\alpha(b))) = \beta'(b)(1 + V'_L(\beta(b))).\]  
(86)

By Lemma 3 \(-1 + V'_L(\alpha(b)) > 0\). By condition (M) \((1 + V'_L(\beta(b))) > 0\). Therefore (86) indicates that: \(\alpha'(b) > 0\) if and only if \(\beta'(b) > 0\). By (82) if \(\alpha'(b) \leq 0\) we must have: \(1 - F_I(b) \leq 0\), which is impossible. \(\square\)

**Lemma 20.** Suppose that the next-period value functions satisfy (C) and (M). When the low type manipulates, we have: \(F_I(b) < P(b)\) and \(P'(b) > 0\).

**Proof of Lemma 20.** Suppose that the low type manipulates. Then we have: \(\alpha(b) + \beta(b) = V_L(\alpha(b)) - V_L(\beta(b))\). By Lemma 18 \(\alpha(b) - \beta(b)\) is non-zero. By dividing both sides by \(\alpha(b) - \beta(b)\), we obtain:

\[\frac{\alpha(b) + \beta(b)}{\alpha(b) - \beta(b)} = \frac{V_L(\alpha(b)) - V_L(\beta(b))}{\alpha(b) - \beta(b)}.\]  
(87)

Since \(\alpha\) and \(\beta\) are increasing and \(V_L\) is strictly convex, we must have:

\[\frac{d}{db} \frac{\alpha(b) + \beta(b)}{\alpha(b) - \beta(b)} = 2 \cdot \frac{\alpha(b)\beta'(b) - \alpha'(b)\beta(b)}{(\alpha(b) - \beta(b))^2} > 0.\]  
(88)

Therefore we conclude:

\[\frac{\beta'(b)}{\beta(b)} > \frac{\alpha'(b)}{\alpha(b)}.\]  
(89)

Notice that:

\[\alpha'(b) \cdot b = \alpha(b) \cdot \frac{F_I(b)}{P(b)} \text{ and } \beta'(b) \cdot b = \beta(b) \cdot \frac{1 - F_I(b)}{1 - P(b)}.\]  
(90)
By (89) we obtain: \( \frac{1 - F_l(b)}{1 - P(b)} > \frac{F_l(b)}{P(b)} \). Since \( F_l(b) > 0 \) and \( P(b) > 0 \), we obtain: \( F_l(b) < P(b) \). We can rewrite this as:
\[
l(b) - l'(b)b(1 - b) - \bar{h} \cdot b + l(b) \cdot (1 - b).
\]
(91)

Thus we conclude: \( 0 < \bar{h} \cdot b - l(b) \cdot b + l'(b)b(1 - b) = P'(b) \cdot b \).

On the other hand, instead suppose that the high-type manipulates. Then the low-type does not manipulate and so \( l(b) \) is constant, which we denote by \( \bar{l} \). Let us define: \( F_h(b) = h(b) + h'(b)b(1 - b) \). Then we can rewrite:
\[
\alpha'(b) = \frac{\bar{l} \cdot F_h(b)}{P(b)^2} \quad \text{and} \quad \beta'(b) = \frac{(1 - \bar{l}) \cdot (1 - F_h(b))}{(1 - P(b))^2},
\]
(92)

and
\[
\alpha''(b) = \frac{\bar{l} \cdot (F_h'(b) \cdot P(b) - 2F_h(b) \cdot P'(b))}{P(b)^3};
\]
\[
\beta''(b) = \frac{(1 - \bar{l}) \cdot (-F_h'(b) \cdot (1 - P(b)) + 2(1 - F_h(b)) \cdot P'(b))}{(1 - P(b))^3}.
\]
(93)

**Lemma 21.** Suppose that the next-period value functions satisfy (C) and (M). When the high-type manipulates, we have: \( F_h(b) > P(b) \) and \( P'(b) > 0 \).

**Proof of Lemma 21.** Suppose that the high type manipulates. Then we have: \( \alpha(b) + \beta(b) - 2 = V_H(\alpha(b)) - V_H(\beta(b)) \). By dividing both sides by \( \alpha(b) - \beta(b)(\neq 0 \) by Lemma 18), we obtain:
\[
\alpha(b) + \beta(b) - 2 = \frac{V_H(\alpha(b)) - V_H(\beta(b))}{\alpha(b) - \beta(b)}.
\]
(95)

Since \( \alpha \) and \( \beta \) are increasing and \( V_H \) is strictly convex, we must have:
\[
\frac{\alpha'(b) + \beta'(b) - 2}{\alpha(b) - \beta(b)} = 2 \cdot \frac{\alpha(b)\beta'(b) - \alpha'(b)\beta(b) + (\alpha'(b) - \beta'(b))}{(\alpha(b) - \beta(b))^2} > 0.
\]
(96)

Since by Lemma 18 \( \beta(b) \neq 1 \) and \( \alpha(b) \neq 1 \), we conclude:
\[
\frac{\beta'(b)}{1 - \beta(b)} < \frac{\alpha'(b)}{1 - \alpha(b)}.
\]
(97)

Notice that:
\[
\alpha'(b) \cdot b = \alpha(b) \cdot \frac{\bar{l}}{h(b)} \cdot \frac{F_h(b)}{P(b)} \quad \text{and} \quad \beta'(b) \cdot b = \beta(b) \cdot \frac{1 - \bar{l}}{1 - h(b)} \cdot \frac{1 - F_h(b)}{1 - P(b)}.
\]
(98)

By (97) we obtain:
\[
\frac{\beta(b)}{1 - \beta(b)} \cdot \frac{1 - \bar{l}}{1 - h(b)} \cdot \frac{1 - F_h(b)}{1 - P(b)} < \frac{\alpha(b)}{1 - \alpha(b)} \cdot \frac{\bar{l}}{h(b)} \cdot \frac{F_h(b)}{P(b)}.
\]
(99)
Thus we obtain:

\[
\frac{\beta(b)}{\alpha(b)} \cdot \frac{1 - \alpha(b)}{1 - \beta(b)} \cdot \frac{1 - \bar{\ell}}{1} \cdot \frac{h(b)}{1 - h(b)} \leq \frac{1 - P(b)}{1 - F_h(b)} \cdot \frac{F_h(b)}{P(b)}.
\]  (100)

By the Bayes rule, we have: \( \frac{\beta(b)}{\alpha(b)} = \frac{1 - h(b)}{h(b)} \cdot \frac{P(b)}{1 - P(b)} \) and \( \frac{1 - \alpha(b)}{1 - \beta(b)} = \frac{\bar{\ell}(1-b)}{\bar{\ell}(1-b)} \cdot \frac{P(b)}{1 - P(b)} \). Thus we obtain:

\[
\frac{\beta(b)}{\alpha(b)} \cdot \frac{1 - \alpha(b)}{1 - \beta(b)} \cdot \frac{1 - \bar{\ell}}{1} \cdot \frac{h(b)}{1 - h(b)} = \frac{1 - h(b)}{h(b)} \cdot \frac{P(b)}{1 - P(b)} \cdot \frac{\bar{\ell}}{1 - \bar{\ell}} \cdot \frac{1 - \bar{\ell}}{1 - h(b)} = 1.
\]

Hence we obtain: \( \frac{1 - P(b)}{1 - F_h(b)} \cdot \frac{F_h(b)}{P(b)} > 1 \). Finally we obtain: \( F_h(b) > P(b) \). We can rewrite this as:

\[
h(b) + h'(b)(1-b) > h(b) \cdot b + \bar{\ell} \cdot (1-b).
\]  (101)

Thus we conclude: \( 0 < h(b) \cdot (1-b) - \bar{\ell} \cdot (1-b) + h'(b)(1-b) = P''(b) \cdot (1-b) \).

\[\text{Lemma 22.} \quad \text{Suppose that the next-period value functions satisfy (C) and (M). When the low-type manipulates at prior } b \in (0, 1), \text{ then it cannot be the case that } \alpha''(b) < 0 \text{ and } \beta''(b) < 0.\]

\[\text{Proof of Lemma 22.} \quad \text{On the contrary, suppose that there is } b \text{ such that } \alpha''(b) < 0 \text{ and } \beta''(b) < 0. \text{ Then, by (83) and (84) we have:}
\]

\[
F'_h(b) < \frac{2F_l(b) \times P'(b)}{P(b)} \quad \text{and} \quad \frac{2(1 - F_l(b))P'(b)}{(1 - P(b))} < F'_l(b).
\]  (102)

Therefore we have:

\[
\frac{(1 - F_l(b))P'(b)}{(1 - P(b))} < \frac{F_l(b) \times P'(b)}{P(b)}.
\]  (103)

Thus we have:

\[
\frac{(1 - F_l(b))}{F_l(b)} < \frac{(1 - P(b))}{P(b)},
\]  (104)

This gives us \( P(b) < F_l(b) \), which is a contradiction by Lemma 20.

Suppose that the low-type manipulates at \( b \). By the continuity of \( \mathcal{E} \) we can take \( m \) sufficiently close to 1 so that the low-type manipulates at \( mb \) and \( b \). For \( b \in R_L \cap R_0 \)

\[
\alpha(b) = \frac{\bar{h}}{P(b)} \cdot b \quad \text{and} \quad \beta(b) = \frac{1 - \bar{h}}{1 - P(b)} \cdot b,
\]

and for \( b \in R_H \cap R_0 \),

\[
\alpha(b) = \frac{\bar{h}}{P(b)} \cdot b \quad \text{and} \quad \beta(b) = \frac{1 - \bar{h}}{1 - P(b)} \cdot b,
\]

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Lemma 23. Suppose that the next-period value functions satisfy (C) and (M). Then $\alpha(b)$ is arc-concave and $\beta(b)$ is arc-convex.

Proof of Lemma 23. Since both bid and ask prices are increasing, we have:

$$
\frac{\bar{h}}{P(b)} \cdot b < \frac{\bar{h}}{P(mb)} \cdot mb \quad \text{and} \quad \frac{1 - \bar{h}}{1 - P(b)} \cdot b < \frac{1 - \bar{h}}{1 - P(mb)} \cdot mb.
$$

Therefore we obtain the desired result. In case of $m < 1$, the proof is done similarly.

Lemma 24. When the low-type manipulates, we have:

$$
\left[\beta(b) + V_L(\beta(b))\right]'' = \left[-\alpha(b) + V_L(\alpha(b))\right]'' \geq 0.
$$

Proof of Lemma 24. On the contrary, suppose not. Then there is a point $b$ such that:

$$
\beta''(b) + [V_L(\beta(b))]'' = -\alpha''(b) + [V_L(\alpha(b))]'' < 0.
$$

First consider the part of ask price $\alpha(b)$. We find that:

$$
(\alpha'(b))^2 \cdot V_L''(\alpha(b)) + \alpha''(b) \cdot V_L'(\alpha(b)) < \alpha''(b).
$$

Since $(\alpha'(b))^2 \cdot V_L''(\alpha(b)) > 0$ and $V_L'(\alpha(b)) > 0$, we have:

$$
\alpha''(b) < 0.
$$

Next we consider the part of bid price $\beta(b)$. We find that:

$$
(\beta'(b))^2 \cdot V_L''(\beta(b)) + \beta''(b) \cdot V_L'(\beta(b)) < -\beta''(b).
$$

Since $(\beta'(b))^2 \cdot V_L''(\beta(b)) > 0$ and $V_L'(\beta(b)) > 0$, we have:

$$
\beta''(b) < 0.
$$

Since at least one of bid and ask prices is convex, this is a contradiction.

6 Uniqueness of Equilibrium

In this section, we make one assumption to prove the uniqueness of equilibrium:

- the informed trading $\mu$ is sufficiently large.

Lemma 25. The value function $W_L$ is arc-convex if and only if the value function $\tilde{W}_H$ is arc-convex.

Proof of Lemma 25. Done by the construction.

Proposition 9. The current period value function $W_H$ is monotonically decreasing and $W_L$ is monotonically increasing in $b$. 

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Proof of Proposition 9. · When \( t = T \)
Since this is the last chance to trade, both types trade on their information. Therefore,
\[
W_H(b) = (1 - \frac{[\mu + (1 - \mu)\gamma]b}{(1 - \mu)\gamma + \mu b}) \cdot \mu = \frac{(1 - b)(1 - \mu)\gamma}{(1 - \mu)\gamma + \mu b} \cdot \mu.
\]
(110)

Therefore, we conclude that \( W_H \) is strictly decreasing in \( b \). \( \square \)

· When \( t = 1, \ldots, T - 1 \)
By Proposition 6, equilibrium exists uniquely. Let \( b > b' \), and \( \sigma' \) denotes the equilibrium strategy when the belief is \( b' \). Then, we have:
\[
W_H(b) = \mu \left( 1 - \alpha(b) + V_H(\alpha(b)) \right) + \left( 1 - \mu \right) \left( \gamma V_H(\alpha(b)) + (1 - \gamma) V_H(\beta(b)) \right)
\]
\[
< \mu \left( 1 - \alpha(b') + V_H(\alpha(b')) \right) + \left( 1 - \mu \right) \left( \gamma V_H(\alpha(b')) + (1 - \gamma) V_H(\beta(b')) \right)
\]
\[
= W_H(b').
\]

Similarly the case for \( W_L \) is proved. This completes our proof. \( \square \)

Proposition 10. There exists a \( m_L \) such that the current-period value function \( W_L \) satisfies (A) and (S).

Proof of Theorem 10. · When \( t = T \)
Since this is the last chance to trade, both types trade on their information. Therefore,
\[
W_L(b) = \frac{b(1 - \mu)(1 - \gamma)}{(1 - \mu)(1 - \gamma) + \mu(1 - b)} \cdot \mu.
\]
(111)

By taking the second derivative we can see that \( W''_L(b) > 0 \) and so \( W_L \) is strictly convex. Therefore, we conclude that \( W_L \) is arc-convex by Proposition 1. \( \square \)

· When \( t = 1, \ldots, T - 1 \)
Consider:
\[
W_L(b) = \mu \beta(b) + \tilde{V}_L(\alpha(b)) + (1 - \tilde{t}) V_L(\beta(b)).
\]
(112)

Notice that:
\[
1 < \frac{h(b)}{P(b)} < \frac{\tilde{h}}{\tilde{t}}; \quad \frac{1 - \tilde{h}}{1 - \tilde{t}} < \frac{1 - h(b)}{1 - P(b)} < 1.
\]
(113)

Suppose that \( V_L(b) \) is arc-convex. Now fix \( \bar{b} \) arbitrarily. Take \( b \) with \( b \leq \bar{b} \). Then, we obtain
\[
\frac{W(b)}{b} \cdot b = \frac{\mu \beta(b) + \tilde{V}_L(\alpha(b)) + (1 - \tilde{t}) V_L(\beta(b))}{\tilde{t}} \cdot b
\]
\[
= \left( \frac{\mu \beta(b)}{\bar{b}} + \tilde{V}_L(\alpha(b)) \cdot \frac{\alpha(b)}{\bar{b}} \right) \cdot \frac{(1 - \tilde{t}) V_L(\beta(b))}{\tilde{t}} \cdot \frac{\beta(b)}{\bar{b}} \cdot \bar{b} \cdot b
\]
(114)

Now take the second derivative of \( W_L \) and then we obtain
\[
W''_L(b) = \\
\mu \beta''(b) + \tilde{t} \left( V''_L(\alpha(b))(\alpha'(b))^2 + V'_L(\alpha(b))\alpha''(b) \right) + (1 - \tilde{t}) \left( V''_L(\beta(b))(\beta'(b))^2 + V'_L(\beta(b))\beta''(b) \right).
\]
(115)
Consider the region of $[b_H, b_L]$. In this region nobody manipulates. So $\beta$ is strictly convex. Moreover, in the region of $[b_L, 1]$, by Lemma 24 we obtain the desired result. This completes our proof. \hfill \qed

**Lemma 26.** Suppose that (C) and (M) hold. All the equilibria hold the Markov property.

**Proof of Lemma 26.** Suppose not. Then, there are two distinct histories $h^t$ and $\hat{h}^t$ and equilibrium strategies $\sigma(h^t)$ and $\hat{\sigma}(\hat{h}^t)$ with $\delta(h^t) = \delta(\hat{h}^t) = b$ for arbitrary $b$. However, this is a contradiction because if (C) and (M) hold, the equilibrium exists uniquely by Proposition 6. \hfill \qed

**Theorem 2.** Equilibrium exists uniquely.

**Proof of Theorem 2.** By Theorem 1, there exists an equilibrium. By Lemma 26, Proposition 6, Proposition 9 and Proposition 10, we conclude that equilibrium exists uniquely. \hfill \qed

**References**


