Rational Bubbles and Macroeconomic Fluctuations

The (De-)Stabilizing Role of Monetary Policy

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Rational bubbles and macroeconomic fluctuations: the (de-)stabilizing role of monetary policy

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March 23, 2012

Abstract

We are interested in the occurrence of expectation-driven fluctuations of a rational bubble and the (de-)stabilizing role of monetary policy. Our explanation of fluctuations is based on credit market imperfections. For this purpose we consider an overlapping generations exchange economy with two assets: a bubble and money needed to finance a share of consumption. Collaterals play a role: holding more bubble reduces this share increasing consumption by credit. Under these credit market features, expectation-driven fluctuations and multiplicity of steady-states occur, especially for arbitrarily low market distortions. Investing stabilizing monetary policy, we show that when the monetary policy rule depends on expected inflation only, a more active rule stabilizes only if collaterals have a strong effect. Finally, we enrich this rule by including asset prices. A policy which depends on asset prices can stabilize whatever the role of collaterals and can also rule out the multiplicity of steady states. More generally, this paper emphasizes the key role of consumers credit market imperfections to explain bubble fluctuations and exhibits the stabilizing power of monetary rules including asset prices.

JEL classification: D91, E32, E52.

Keywords: Rational bubble; Cash-in-advance constraint; Collaterals; Endogenous fluctuations; Monetary policy.

*We would like to thank Edouard Challe, Jean-Michel Grandmont, Takashi Kamihigashi, Hubert Kempf, Olivier Loisel, Carine Nourry, Yiannis Vailakis, Alain Venditti and Bertrand Wigniolle for useful suggestions. We also thank participants to the Conference in honor of Cuong Le Van held in University Paris 1 on December 2011. Any remaining errors are our own.

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1 Introduction

Over the last decades, financial markets have frequently gone through periods of excessive asset price volatility. Some empirical studies, Shiller (1981, 1989, 2000), LeRoy and Porter (1981), and more recently Campbell (2003), highlight this excess volatility and reveal that asset prices tend to fluctuate more than the underlying fundamentals. These contributions reflect the idea that the excess volatility could come from fluctuations of a bubble.

In the collective consciousness, and many political debates, the fluctuations of a bubble are often associated to the irrational behavior of agents. In 1996, Alan Greenspan uses the expression “irrational exuberance” to describe the movements of speculative bubbles in financial markets.\(^1\) In this paper, we provide an alternative justification of such fluctuations, which lies in the volatility of rational expectations, i.e. in the existence of persistent expectation-driven fluctuations of a rational bubble.

Besides, asset prices are in the heart of another debate, namely whether central banks should take or not into account asset price movements in monetary policy. As for academic research, the debate is not resolved: among others, Bernanke and Gertler (2001), Gilchrist and Leahy (2002), Carlstrom and Fuerst (2007) and Gali (2011) provide a negative answer about including asset prices in monetary rules, whereas Nutahara (2010) and Singh and Stone (2011) develop models in favor of it. Moreover, several Governors of central bank or practitioners of monetary policy recently encourage more research on these topics. In his recent speech, Bernanke (2011) argues that: “In my view, the issue is not whether central bankers should ignore possible financial imbalances – they should not – but, rather, what “the right tool for the job” is to respond to such imbalances.” In our framework where fluctuations are explained by the volatility of expectations, we also contribute to this debate: a monetary policy which responds to movements in asset prices stabilizes economic fluctuations, whereas this is not always the case when the rule depends on inflation forecasts only.

We build a simple general equilibrium model in which rational bubbles could experience persistent fluctuations, and allow us to study the (de-)stabilizing role of monetary policy. Our explanation is mainly based on credit market imperfections and the role of collaterals. In contrast to recent contributions studying the link between rational bubbles and credit market imperfections faced by entrepreneurs (Farhi and Tirole (2010), Kocherlakota (2009)), but more in line with Kocherlakota (1992) or Hellwig and Lorenzoni (2009), we focus on consumers credit constraint.\(^2\) We introduce such a constraint in a monetary overlapping generations model. Indeed, following Tirole’s seminal papers (1982, 1985), this type of models provides a simple and useful framework to deal with the existence of rational bubbles.\(^3\)

\(^1\)This expression is also taken up by Robert Shiller (2000) as the title of his book.

\(^2\)Some empirical studies highlight financing constraints at the household level. See for instance Campbell and Mankiw (1989).

\(^3\)However, the existence of rational bubbles could also be provided in infinitely-lived household models. For instance, Kocherlakota (1992) develop a model with heterogeneous agents
The issue of fluctuations of a rational bubble within an overlapping generation model has been addressed only in few works (Grandmont (1985), Weil (1987), Michel and Wigniolle (2003, 2005), Bosi and Seegmuller (2010) and Wigniolle (2012)). Our aim is to provide an additional explanation of expectation-driven fluctuations of a bubble based on credit market imperfections, which create a portfolio effect, and to study the (de-)stabilizing role of monetary rules including inflation forecast and asset prices.\(^4\)

We consider an exchange economy, in which households realize a portfolio choice, holding an asset without fundamental value (the bubble) and money, needed for transactions.\(^5\) A share of second-period consumption is paid by cash, while the rest is financed through non-monetary savings, i.e. by credit.\(^6\) This cash-in-advance constraint is introduced here as a credit market imperfection.

One important feature is that the share of consumption purchased on credit depends positively on the value of the bubble held by agents. This allows us to introduce the notion of collaterals: if agents hold more bubble assets, they can increase their opportunity to obtain credits and their share of consumption purchased on credit. This assumption means also that distortions on credit market decrease with collaterals. Moreover, since we focus on a binding cash-in-advance constraint, the portfolio choice is endogenously determined and variable over time, depending on expectations about future asset prices.

Our main results lean on credit market features. A persistent rational bubble experience fluctuations due to the volatility of expectations under slightly restrictive conditions, i.e. for arbitrarily small credit market imperfections and for relatively low income effects. This result answers a recurrent criticism about endogenous fluctuations in monetary overlapping generations models. The intuition is the following. As collaterals matter, a change in the portfolio structure occurs. A rise in expected inflation generates a portfolio effect: agents reallocate savings from the monetary savings to the bubble. Households consume less by cash in the second period of life, the return on money decreases. An effective rise of inflation takes place, i.e. expectations are self-fulfilling. In connection with this result, we show that the multiplicity of steady states occurs under the same conditions: convergence or not to a steady state may depend on agents’ expectations. Furthermore, we obtain also the well-known result obtained by Grandmont (1985, 1986). Expectation-driven fluctuations and two-period cycles occur under large income effects, i.e. when savings are an increasing function of expected inflation whatever the level of credit market imperfections.

\(^4\)Hence, even if we do not consider capital accumulation, we generalize the analysis provided in Bosi and Seegmuller (2010) in two directions. First, we provide a more deeply analysis of dynamics and second, we consider monetary policies not only characterized by a constant growth rate.

\(^5\)In contrast to Michel and Wigniolle (2003, 2005), we focus on equilibria where the cash-in-advance constraint is always binding. As a consequence, bubbles on real money balances cannot exist.

\(^6\)See Hahn and Solow (1995). For further details about cash-in-advance constraints in overlapping generations model, the reader can refer to Crettez et al. (1999).
We pursue by analyzing the stabilizing role of monetary policy on the fluctuations of the rational bubble. The question of whether monetary policy can stabilize endogenous fluctuations has been essentially studied in models without portfolio choice and without collaterals (Grandmont (1985, 1986), Sorger (2005)). While Grandmont (1985, 1986) suggests monetary policies which are able to coordinate expectations, Sorger (2005) proposes to study monetary rules, based on inflation forecast targeting, that alter conditions for indeterminacy. He shows that an active rule, i.e. a monetary rule which strongly reacts to expected inflation, renders the economy more susceptible to expectation-driven fluctuations, increasing the range of parameters for indeterminacy. We show that the presence of collaterals can reverse this established result: an active inflation forecast targeting rule could locally stabilize as soon as effects of collaterals on the credit share are strong enough. However, inflation forecast targeting does not have any impact on fluctuations which occur for low income effects, and does not modify conditions under which global indeterminacy may occur.

Considering a monetary rule that responds to movements in asset prices seems to be appropriate to stabilize an economy with fluctuations of a bubble. We extend the rule of inflation targeting proposed by Sorger (2005), by including asset prices directly. A monetary rule with asset prices could locally and globally stabilize fluctuations unlike a monetary rule which responds only to inflation. More precisely, when collaterals have weak effects on the credit share, a passive monetary rule should be led, and when collaterals matter sufficiently, an active one should be led. Following the debate initiated by Bernanke and Gertler (1999, 2001) and recently advanced by several Governors of central bank or practitioners of monetary policy (Yellen (2009), Bernanke (2010, 2011)), we provide a clear-cut conclusions. A monetary policy including asset prices is powerful to stabilize expectation-driven fluctuations.

The rest of the paper is organized as follows. The Section 2 is devoted to the presentation of the model. In Section 3, the intertemporal equilibrium is defined. Steady states with a bubble are studied in Section 4. In Section 5, we show the existence of bubble fluctuations. Section 6 discusses the (de-)stabilizing role of monetary policies. A last section provides concluding remarks, whereas some technical details are relegated to an Appendix.

2 The model

We focus on an overlapping generations exchange economy with identical two period-lived households, discrete time \( t = 0, 1, ..., +\infty \) and three goods: a final good, money and an asset paper.

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7 We adopt the terminology initiated by Benhabib et al. (2001).

8 Note that in a different framework, Benhabib et al. (2001) provide an alternative argument against inflation forecast targeting rule, based on global indeterminacy. In a recent speech, Bernanke (2010) explains however that the US monetary policy follows more closely such a rule than one based on observed inflation.
2.1 Households

There is no population growth and, at each date \( t \), a generation of unit size is born. Each generation lives two periods.

In their first period of life, households are endowed with \( e \geq 0 \) units of consumption good and receive a monetary transfer \( T_t \) from the monetary authority that they allocate between the purchase of the consumption good \( c_t \) at price \( p_t \) and savings, in the form of nominal balances \( M_{t+1} \) and an asset paper without fundamental value. Let \( B_t \) be the price of this asset and \( 1 + i_{t+1} \) its return factor between \( t \) and \( t + 1 \). In their second period of life, since they have neither endowments nor monetary transfer, households use their remunerated savings \( M_{t+1} + (1 + i_{t+1}) B_t \) to consume \( d_{t+1} \) at price \( p_{t+1} \).

The preferences of an household are represented by an additively separable life-cycle utility function:

\[
u(c_t) + \beta v(d_{t+1})
\]

with \( \beta > 0 \).

**Assumption 1** \( u(c) \) and \( v(d) \) are defined on \( \mathbb{R}^2 \), \( C^2 \) on \( \mathbb{R}^2_+ \), strictly increasing \( (u' > 0, v' > 0) \) and concave \( (u'' < 0, v'' < 0) \). Moreover, \( \lim_{c \to 0} u'(c) = +\infty \) and \( \lim_{d \to 0} v'(d) = +\infty \). We define \( \epsilon_u(c) \equiv -c u''(c)/u'(c) \) and \( \epsilon_v(d) \equiv -d v''(d)/v'(d) \) as the degrees of concavity of \( u(c) \) and \( v(d) \) respectively.

Under perfect foresight, the representative household of a generation born at time \( t \) derives consumption plan and asset demands (money and asset paper) by maximizing the utility function (1) under the first and second-period budget constraints:

\[
\begin{align*}
p_t c_t + M_{t+1} + B_t &\leq p_t e + T_t \quad (2) \\
p_{t+1} d_{t+1} &\leq M_{t+1} + (1 + i_{t+1}) B_t \quad (3)
\end{align*}
\]

Furthermore, at the second period of life, each household faces a cash-in-advance constraint. We use the constraint introduced by Hahn and Solow (1995), i.e. \( \gamma p_{t+1} d_{t+1} \leq M_{t+1} \), but extend it to take into account collaterals:

\[
\gamma (b_t) p_{t+1} d_{t+1} \leq M_{t+1} \quad (4)
\]

where \( b_t \equiv B_t/p_t \) is real non-monetary savings.

When the constraint is binding, a share \( \gamma (b_t) \in (0, 1) \) of consumption purchases has to be paid cash, while the remaining share \( 1 - \gamma (b_t) \) can be paid from the resale of the asset papers in the financial markets and refers to the credit share. Holding asset papers enables the household to borrow an amount equal to \( (1 + i_{t+1}) B_t \) to finance a share of the second-period consumption, whereas these asset papers will be resold only at the end of the second period.\(^9\)

\(^9\)A similar idea can be seen in Lucas and Stokey (1987), where consumption purchased with money would correspond to “cash goods” and consumption purchased on credit to “credit goods”. 
In addition, we assume that the credit share depends positively on the amount of non-monetary savings for two reasons: the velocity of money and the credit market frictions. First, we give an answer to a recurrent criticism addressed to the cash-in-advance literature: money velocity $1/\gamma(b)$ is endogenous and no longer constant. Second, some financial frictions in borrower-lender relationship as asymmetric information or financial regulation policy are introduced through this assumption. The non-monetary savings (asset papers) work as collaterals: a household who owns more asset papers can increase his opportunities to obtain credit and reduces his need of cash in his second-period of life.

**Assumption 2** $\gamma(b) \in (0, 1)$ is a continuous function defined on $[0, +\infty)$, $C^2$ on $(0, +\infty)$, decreasing ($\gamma'(b) \leq 0$) and such that $\gamma(0) < 1 - \epsilon$, where $\epsilon$ is sufficiently close to zero. In addition, we define:

$$\eta_1(b) \equiv \frac{[1 - \gamma(b)]' b}{1 - \gamma(b)} \geq 0, \quad \eta_2(b) \equiv -\frac{[1 - \gamma(b)]'' b}{[1 - \gamma(b)]'}$$

$$\eta_\eta(b) \equiv \frac{\eta_1'(b) b}{\eta_1(b)} = 1 - \eta_1(b) - \eta_2(b)$$

Notice that when collaterals play no role ($\eta_1(b) = 0$), and $\gamma$ tends to 0, money is no longer needed and the credit market distortion disappears. When collaterals matter ($\eta_1(b) > 0$), the households are aware of the credit share function so that they take into account its argument $b$ in their decisions.

Using $\pi_{t+1} \equiv p_{t+1}/p_t$ and introducing the real variables $m_t \equiv M_t/p_t$ and $\tau_t \equiv T_t/p_t$, the constraints (2)-(4) can be rewritten:

$$c_t + \pi_{t+1}m_{t+1} + b_t \leq e + \tau_t$$

$$d_{t+1} \leq m_{t+1} + \frac{1 + i_{t+1} b_t}{\pi_{t+1}}$$

$$\gamma(b_t) d_{t+1} \leq m_{t+1}$$

The representative household maximizes (1) under the budget and cash-in-advance constraints (7)-(9) to determine optimal consumption plan ($c_t, d_{t+1}$) and optimal savings ($m_{t+1}, b_t$). All constraints are binding if money is a dominated asset ($(1 + i_{t+1})/\pi_{t+1} > 1/\pi_{t+1}$), or equivalently the opportunity cost of holding money, the nominal interest rate $i_{t+1}$, is strictly positive ($i_{t+1} > 0$). Only a bubble on asset papers could appear.

In the remaining of the paper, we only consider the case of a binding cash-in-advance constraint so that the portfolio choice is determined. The asset paper holding, whose level depends on expectations on its future value, determines the money demand used for future consumption. Nonetheless, the endogeneity of the credit share allows the portfolio choice to be no longer constant. The trade-off between assets becomes endogenous and depends on the amount of collaterals hold by the households, implying the existence of a portfolio effect. This effect could be seen as a substitution effect within the portfolio following
a change in the relative price between the two assets. Thereafter, we will see that this portfolio effect is a key mechanism through which expectation-driven fluctuations emerge.

**Assumption 3** For all $t \geq 0$, we assume $i_t > 0$ and

$$\eta_1 (b_t) < \frac{\gamma (b_t) b_t}{1 - \gamma (b_t) \pi_{t+1} m_{t+1}}$$

(10)

**Lemma 1** Under Assumption 3, constraints (7)-(9) are binding.

**Proof.** See the Appendix.

Inequality (10) puts an upper bound to the credit-share elasticity $\eta_1 (b)$. If collaterals matter too much, the agents no longer hold money in their portfolio. As a result, the cash-in-advance constraint fails to be binding. Inequality (10) is specific to our model because of the role of collaterals. Since the right-hand side is strictly positive, it is satisfied when the credit share $1 - \gamma$ is constant ($\eta_1 = 0$). Thereafter, we focus on the case where $\eta_1$ is not too strong. In other words, we consider small distortions on the credit market.

Under Assumption 3, the optimal households’ behavior is summarized by the following equation:

$$\frac{u' (c_t)}{\beta v' (d_{t+1})} = \frac{1}{\pi_{t+1}} \frac{(1 + i_t) / \pi_{t+1} + \gamma (b_t) d_{t+1}}{(1 - \gamma) / \pi_{t+1} + \gamma (b_t) d_{t+1}}$$

(11)

where the last inequality holds because money is a dominated asset $((1 + i_t) / \pi_{t+1} > 1 / \pi_{t+1})$. Furthermore, we deduce the following lemma$^{10}$:

**Lemma 2** Let

$$\tilde{\varepsilon}_u \equiv c \frac{(1 - \gamma)^2}{[1 - \gamma + (1 + i_t) (\gamma - \eta_1)]} \pi^2 \eta_1 \left(2 - \frac{\eta_2}{1 - \eta_1}\right)$$

where $\tilde{\varepsilon}_u > 0$ if and only if $\eta_2 < 2 (1 - \eta_1)$. The second-order conditions are satisfied if $\varepsilon_u > \tilde{\varepsilon}_u$.

**Proof.** See the Appendix.

We further note that under a constant credit share $(1 - \gamma (b) = 1 - \gamma)$, equation (11) rewrites:

$$\frac{u' (c_t)}{\beta v' (d_{t+1})} = \frac{1 + i_t + i_t + i_t / \pi_{t+1}}{1 + i_t + i_t}$$

(12)

When $\gamma$ tends to 0, the right-hand-side would reduce to $(1 + i_t + i_t) / \pi_{t+1}$, which is similar to the trade-off found in the monetary model by Samuelson (1958).

$^{10}$For simplicity, the arguments of the functions and the time subscripts are omitted.
There is no market distortions. When $\gamma > 0$, money demand implies an opportunity cost which lowers the real return on portfolio. More precisely, the household has to pay by cash a share $\gamma$ to consume an extra-unit when he is old. The interest rate $i_{t+1}$ entails an opportunity cost $\gamma i_{t+1}$ which reduces the purchasing power of savings in the form of asset papers. Furthermore, when the credit share depends on collaterals, the marginal effect of non-monetary savings on the credit share $(-\gamma'(b) > 0)$ becomes an additional distortion.

### 2.2 Monetary rule

Let us introduce the money growth factor $\mu_t = M_{t+1}/M_t$. It can be rewritten:

$$\mu_t = \frac{\pi_{t+1}}{\mu_t}$$

(13)

As in Sorger (2005), it corresponds to the monetary instrument. However, focusing on bubble fluctuations, we extend the money growth rate rule proposed by Sorger (2005), by taking deviations of observed asset price from the target into account. Although the instrument of the rule is the money growth rate, our formulation corresponds to the one suggested by Bernanke and Gertler (1999, 2001), Carlstrom and Fuerst (2007) and Singh and Stone (2010).

The monetary policy is implemented through an instrument rule of the following form:

$$\mu_t = \mu^* \left( \frac{\pi_{t+1}}{\mu^*} \right)^{-\alpha} \left( \frac{b_t}{b^*} \right)^{\rho}$$

(14)

with $\alpha \in (-1, +\infty)$ and $\rho \in (-1, +\infty)$.

We clarify that $\mu^*$ and $b^*$ are respectively the stationary values of the money growth factor and the asset price level of an existing stationary equilibrium chosen as the target by the monetary authority.

When $\alpha = 0$ and $\rho = 0$, the nominal money supply grows at the constant growth factor $\mu_t = \mu^*$. This policy corresponds to a regime of strict money growth targeting in which the nominal money growth rate is fixed at its stationary value $\mu^*$.

When $\alpha \neq 0$ and $\rho = 0$, equation (14) depicts a policy rule for inflation forecast targeting (Sorger (2005)). When the monetary authority sets the money growth factor $\mu_t$, the inflation $\pi_{t+1}$ has not yet been observed and can be seen as an inflation forecast. For $\alpha > 0$, the nominal money growth is a decreasing function of expected inflation. The nominal money growth is contracted if expected inflation is above its stationary value $\mu^*$ and expanded if expected inflation is below. For $\alpha \in (-1, 0)$, the nominal money growth is an increasing function of expected inflation but less than proportional. Following Benhabib et al. (2001), we call a rule with $\alpha > 0$ an active one and a rule with $\alpha \in (-1, 0)$ a passive one.

When $\alpha \neq 0$ and $\rho \neq 0$, equation (14) depicts a policy rule which takes into account the level of asset prices. The policy rule includes a response to the level of current asset prices $b_t$ with respect to its steady state level $b^*$.
This rule is in accordance with Bernanke and Gertler (1999, 2001). In addition, several Governors of central bank or practitioners of monetary policy encourage more research on monetary policies which would react to credit booms (Yellen (2009)). By considering \( b_t \) as savings in the form of credits and thus as an index of the credit level, equation (14) could also been interpreted as a formalisation of such a monetary policy.

For \( \rho > 0 \), the nominal money growth rate is a decreasing function of current asset prices. In other words, the nominal money supply is contracted if the level of asset prices is above \( b^* \) and it is expanded if the level is below. For \( \rho \in (-1, 0) \), the nominal money growth rate is an increasing function of current asset prices but less than proportional. In the following, we call a rule with \( \rho > 0 \) an active one and a rule with \( \rho \in (-1, 0) \) a passive one.

Money is distributed by the monetary authority to young households through a lump-sum transfer \( \tau_t = (M_{t+1} - M_t)/p_t \), or equivalently,

\[
\tau_t = \pi_{t+1}m_{t+1} - m_t
\]  

(15)

Assuming that the monetary transfer is distributed in the first period of life and not in the second period, we closely follow Michel and Wigniolle (2005). This assumption seems to be more appropriate to study the role of savings and the portfolio choice on dynamics. Actually, the monetary transfer distributed in the second period of life negatively affects the amount of individual savings.

2.3 Asset paper

Following Tirole (1985), Weil (1987) and more recently Bosi and Segmuller (2010), we assume that there is an asset paper, without fundamental value, which may be use to save. It is supplied in a constant amount, normalized to one. As already seen, \( B_t \) denotes its monetary price, which grows at the rate \( i_{t+1} \):

\[
B_{t+1} = (1 + i_{t+1})B_t
\]  

(16)

The asset is called a bubble, when its price is nonnegative, i.e. \( B_t > 0 \). Using real variables, equation (16) can be rewritten:

\[
b_{t+1} = \frac{1 + i_{t+1}}{\pi_{t+1}}b_t
\]  

(17)

3 Intertemporal equilibrium

Substituting (15) in the first-period budget constraint (7), we find:

\[
c_t + m_t + b_t = e
\]  

(18)

Using (8), (9), (17) and (18), we obtain:

\[
m_{t+1} = b_{t+1} \frac{\gamma(b_t)}{1 - \gamma(b_t)}
\]  

(19)
\[ c_t = e - \frac{b_t}{1 - \gamma (b_{t-1})} \quad (20) \]
\[ d_{t+1} = \frac{b_{t+1}}{1 - \gamma (b_t)} \quad (21) \]

As \( c_t > 0 \), we deduce an upper bound on \( b_t \) from (20):

\[ 0 \leq b_t < [1 - \gamma (b_{t-1})] e \equiv \bar{b} (b_{t-1}) \quad (22) \]

Using (13), (17) and (19), we deduce the inflation factor and the rental rate on asset papers:

\[ \pi_{t+1} = \mu_t \gamma (b_{t-1}) \frac{1 - \gamma (b_t)}{1 - \gamma (b_{t-1})} b_{t+1} \quad (23) \]
\[ 1 + i_{t+1} = \mu_t \gamma (b_{t-1}) \frac{1 - \gamma (b_t)}{1 - \gamma (b_{t-1})} \quad (24) \]

Substituting (18)-(24) into (11), we obtain the consumers’ intertemporal trade-off:

\[ \frac{u'}{\beta v'} \left( e - \frac{b_t}{1 - \gamma (b_{t-1})} \right) = \frac{b_{t+1}}{b_t} \frac{[1 - \eta_t (b_t)]}{1 - \gamma (b_t) + \gamma (b_t) - \eta_t (b_t)} \quad (25) \]

Note that \( b_0 \) is not predetermined in period 0. However, the second-order recurrence equation (25) has a predetermined variable at period \( t \), \( b_{t-1} \). At this stage, it is also relevant to clarify that from Assumption 1 inequality (22) is satisfied.

From (13), (14), and (19), we obtain:

\[ \frac{\pi_{t+1}}{\mu^*} = \left[ \gamma (b_{t-1}) \frac{1 - \gamma (b_t)}{1 - \gamma (b_{t-1})} \frac{b_t}{b_{t+1}} \right]^{\frac{\mu_t}{\beta \nu'}} \left( \frac{b_t}{b^*} \right)^{\frac{-\mu_t}{\beta \nu'}} \quad (26) \]

Substituting (26) into (14), we get the following expression of the money growth rate:

\[ \mu_t = \mu^* \left[ \gamma (b_{t-1}) \frac{1 - \gamma (b_t)}{1 - \gamma (b_{t-1})} \frac{b_t}{b_{t+1}} \right]^{\frac{\mu_t}{\beta \nu'}} \left( \frac{b_t}{b^*} \right)^{\frac{-\mu_t}{\beta \nu'}} \quad (27) \]

**Definition 1** Under Assumptions 1-3, an intertemporal equilibrium with perfect foresight is a sequence \((b_t)\), with \( 0 \leq b_t < \bar{b} (b_{t-1}) \) and \( t = 0, 1, ..., +\infty \), such that (25) is satisfied, where \( \mu_t \) is given by (27).

For further reference, note that substituting (17) and (19) into (10) allows us to rewrite Assumption 3:

\[ 1 < 1 + i_{t+1} < 1/\eta_t (b_t) \quad (28) \]

for \( t = 0, 1, ..., +\infty \), which ensures that the cash-in-advance constraint is binding.
In the following, we will use this definition of the equilibrium to show the existence of expectation-driven fluctuations of the rational bubble and study the (de-)stabilizing role of monetary policy. For this purpose, we first analyze the steady state. We will see that some new interesting results, related to the multiplicity of solutions and based on credit market features, will be obtained.

4 Steady state analysis

A bubbly steady state is a solution solving:

\[
\frac{u'(e - b \frac{b}{1 - \gamma(b)})}{\beta v'(\frac{b}{1 - \gamma(b)})} = \frac{1 - \eta_1(b)}{1 - \gamma(b) + [\gamma(b) - \eta_1(b)]\mu(b)},
\]

(29)

where \(\mu(b) \equiv \mu^*(b/b^*)^{-\rho/(1+\alpha)}\) is obtained from (27).

In the following, we first study the existence of the bubbly steady state. Second, we establish the existence of a normalized steady state. The issue of uniqueness versus multiplicity of stationary solutions is finally addressed.

4.1 Existence and uniqueness

The following assumption helps us to prove the existence of a steady state with a positive bubble:

**Assumption 4** If \(\rho > 0\),

\[
\varepsilon_v(0) < \frac{\eta_1(0)}{[1 - \gamma(0)](1 - \eta_1(0))} \frac{[1 - \gamma(0)][\mu(0) - 1]}{[1 - \gamma(0) + \mu(0)[\gamma(0) - \eta_1(0)]]} + \frac{\rho - \mu(0)[\gamma(0) - \eta_1(0)]}{1 + \alpha (1 - \gamma(0) + \mu(0)[\gamma(0) - \eta_1(0)])}
\]

The next proposition proves the existence of a bubbly steady state and provides a result on uniqueness.

**Proposition 1** Let \(\bar{b}\) be defined by \(e = \bar{b}/[1 - \gamma(\bar{b})]\). Under Assumptions 1-4, there exists a steady state characterized by a positive bubble, \(b^* \in (0, \bar{b})\). The uniqueness of the steady state is ensured if the following condition holds \(\forall b \in (\bar{b}, \bar{b})\), with \(b\) arbitrarily close to zero:

\[
\varepsilon_v(d) > \frac{b}{c(1 - \gamma(b))} (\varepsilon_u^*(b) - \varepsilon_u(c)),
\]

(30)

with

\[
\varepsilon_u^*(b) \equiv \frac{c(1 - \gamma(b))}{\beta} \left\{ \frac{\eta_1(b)}{1 - \gamma(b)} \frac{[1 - \gamma(b)][\mu(b) - 1]}{1 - \gamma(b) + \mu(b)[\gamma(b) - \eta_1(b)]} \right\} + \frac{\rho}{1 + \alpha (1 - \gamma(b) + \mu(b)[\gamma(b) - \eta_1(b)])}
\]

\[
c = e - \frac{b}{1 - \gamma(b)} \quad \text{and} \quad d = \frac{b}{1 - \gamma(b)}
\]
Proof. See the Appendix.

Under a constant credit share $1 - \gamma (\eta_1 = 0)$ and a passive rule on asset prices $(\rho \leq 0)$, the condition (30) is always satisfied. This is also the case when the degrees of concavity of utility function are large enough. However, multiplicity may occur when (30) is not satisfied for all $b \in (b, \bar{b})$. We clarify this in the next section.

Recall that the monetary authority specifies targets for inflation and asset prices, choosing the stationary values of an existing steady state. From Proposition 1, a steady state with bubble $b^*$ always exists. We assume that the monetary authority selects this equilibrium $b^*$ as the target.

4.2 Normalized steady state and multiplicity

In order to facilitate the analysis of the multiplicity of steady states and of local dynamics (Sections 5 and 6), we establish the existence of a normalized steady state $b^* = 1$ (NSS). We follow the procedure introduced by Cazzavillan et al. (1998) and use the scaling parameter $\beta$, to give conditions for the existence of such a steady state.

Proposition 2 Under Assumptions 1-4, there exists a unique value $\beta^* > 0$ given by

$$\beta^* = \frac{u'(c - \frac{1}{1 - \gamma(1)})}{u'\left(\frac{1 - \eta_1(1)}{1 - \gamma(1) + \mu(1)[\gamma(1) - \eta_1(1)]}\right)}$$

such that $b^* = 1$ is a steady state of the dynamic equation (25).

Thereafter, we assume $\beta = \beta^*$ so that $b^* = 1$. At the normalized steady state, $\mu(1) = \mu^* = \pi$ (see (13)) and $\pi = 1 + i$ (see (17)). From (28), Assumption 3 is satisfied if and only if:

$$1 < \mu^* < 1/\eta_1 (1)$$

(31)

Furthermore, the second order conditions are satisfied for $\varepsilon_u > \bar{\varepsilon}_u$, with $\bar{\varepsilon}_u \equiv e^* (1 - \gamma)^\eta_1 (1 - \gamma - \mu^* [\gamma - \eta_1]) / [1 - \gamma + \mu^* (\gamma - \eta_1)]$.

We further note $\varepsilon_u$ and $\varepsilon_v$ the degrees of concavity $\varepsilon_u (c)$ and $\varepsilon_v (d)$ evaluated at the steady state $b^* = 1$, $\varepsilon_u^* \equiv \varepsilon_u^*(1)$, $\varepsilon_v^* \equiv \varepsilon_v(1)$, $\gamma \equiv \gamma (1)$, $\eta_1 \equiv \eta_1 (1)$, $\eta_2 \equiv \eta_2 (1)$ and $\mu (1) = \mu^*$.

We can now clarify the conditions for the multiplicity of steady states.

Proposition 3 Under Assumptions 1-4, if the following condition holds:

$$\varepsilon_v < \frac{1}{c(1 - \gamma)} (\varepsilon_u^* - \varepsilon_u)$$

(32)

there is a multiplicity of stationary equilibria with positive bubble. Their number is generically odd.
Proof. See the Appendix.

When the monetary rule does not respond to asset prices ($\rho = 0$), inequality (32) can be satisfied only if collaterals have weak effect on the credit share ($\eta_1 < \gamma$) and $\varepsilon_u^{*}$ is positive.\footnote{The second order conditions are satisfied if $\varepsilon_u > \bar{\varepsilon}_u$ and $\varepsilon_u^{*} > \bar{\varepsilon}_u$ is satisfied only when $\eta_1 < \gamma$.} Thinking that this is interesting to connect the multiplicity of steady states to local dynamics, we assume for the remainder of the paper:

**Assumption 5**

$$\eta_2 < 2(1 - \eta_1)$$

which ensures that $\varepsilon_u^{*} > 0$.

When monetary rule takes into account asset prices, inequality (32) is satisfied when the degrees of concavity $\varepsilon_v$ and $\varepsilon_u$ are weak enough, when an active rule on asset prices ($\rho > 0$) is implemented and collaterals have a weak effect on the credit share ($\eta_1 < \gamma$) or when a passive rule ($\rho < 0$) on asset prices is implemented and collaterals have a large effect on the credit share ($\eta_1 > \gamma$).

With the multiplicity of steady states, convergence or not to a steady state may depend on agents’ expectations.\footnote{Note that, in our model, this multiplicity will be related to the local indeterminacy.} The existence of multiple steady states may lead to a global indeterminacy, which is a source of expectation-driven fluctuations of the bubble. Interestingly, when the monetary policy rule does not depend on asset price, this multiplicity appears under small distortions, i.e. $\eta_1$ weak with $\eta_1 < \gamma$, and low degrees of concavity on utility.

Furthermore, multiplicity raises an issue for the monetary authority concerning the choice of the rule. If the authority chooses to lead an active policy for the steady state $b^* = 1$, this does not mean that the policy would be active at the other steady states. This issue of selection is addressed for instance by Benhabib et al. (2001). Nonetheless, in our model, multiplicity arises without introducing any bound on the monetary rule.

5 Expectation-driven fluctuations and endogenous cycles

This section is devoted to the existence of expectation-driven fluctuations of a rational bubble. Our explanation mainly lies in credit markets features. We will show that the steady state with a positive bubble can be locally indeterminate and therefore, expectation-driven fluctuations of the bubble can emerge. To highlight bubble fluctuations, we consider the model with a constant money growth, first when the credit share is constant, second when collaterals matter. When the credit share is constant, indeterminacy occurs always under a sufficiently high degree of concavity of utility. In contrast, when it depends on asset
holding, a new interesting result is obtained: indeterminacy also occurs under a sufficiently low degree of utility concavity and arbitrarily weak credit market distortions. In the next section, we enrich the model with monetary rules, based on inflation forecast targeting and on asset prices to study the (de-)stabilizing role of monetary policy.

5.1 Local dynamics: preliminaries

To derive our different results, we start by linearizing the dynamic equation (25) around the steady state \( b^* = 1 \) to obtain the characteristic polynomial.

**Assumption 6**

\[ \varepsilon_v \neq \frac{1 - \gamma + \mu^* (\gamma - \eta_1)}{1 - \gamma + \mu^* (\gamma - \eta_1)} \equiv \bar{\varepsilon}_v \]

Using this assumption, we get:

**Lemma 3** Let

\[ \bar{\varepsilon}_u \equiv -c^* (1 - \gamma) \frac{\mu^* (\gamma - \eta_1)}{1 - \gamma + \mu^* (\gamma - \eta_1) \frac{1 + \alpha}{\gamma}} \]  

Under Assumptions 1-6, the characteristic polynomial, evaluated at the steady state \( b^* = 1 \), is defined by \( P(X) \equiv X^2 - TX + D = 0 \), where:

\[ D = 1 - \frac{\bar{\varepsilon}_v}{\varepsilon_v} \frac{\eta_1}{c^* (1 - \gamma)} \frac{(\varepsilon_u - \bar{\varepsilon}_u)}{1 - \gamma + \mu^* (\gamma - \eta_1)} \equiv D(\varepsilon_v) \]

\[ T = \frac{1}{\bar{\varepsilon}_v - \varepsilon_v} \left\{ \frac{(\varepsilon_u - \bar{\varepsilon}_u)}{c^* (1 - \gamma)} - \eta_1 \varepsilon_v + \frac{(1 - \gamma)}{1 - \gamma + \mu^* (\gamma - \eta_1) \frac{1 + \alpha}{\gamma}} \right\} \equiv T(\varepsilon_v) \]

**Proof.** See the Appendix.

As in Grandmont et al. (1998), we study the variations of the trace \( T(\varepsilon_v) \) and the determinant \( D(\varepsilon_v) \) in the \( (T,D) \) plane as one of the parameters of interest, namely \( \varepsilon_v \) is made to vary continuously in its admissible range \( (0, +\infty) \) (see Figures 1 and 2). The locus \( \Sigma \equiv \{(T(\varepsilon_v), D(\varepsilon_v)) : \varepsilon_v \geq 0\} \) describes a part of a line, that we call the \( \Sigma \)-line.

Along the line \((AC)\), one eigenvalue is equal to 1 \((D = T - 1)\). Along the line \((AB)\), one eigenvalue is equal to \(-1\) \((D = -T - 1)\). Along the segment \([BC]\) \((|T| < 2, D = 1)\), the characteristic roots are complex conjugates with modulus equal to 1. These lines divide the space \((T,D)\) into three different types of regions. Therefore, inside the triangle \(ABC\), the steady state is a sink, i.e. locally indeterminate \((|T| < 1 + D\) and \(D < 1)\). It is a saddle point if \((T,D)\) lies on the right or left sides of both the lines \((AB)\) and \((AC)\) \((|1 + D| < |T|)\). It is a source otherwise.
A (local) bifurcation arises when at least one eigenvalue crosses the unit circle, that is, when the Σ-line crosses one of the loci (AB), (AC) or [BC].

According to the changes of the bifurcation parameter, a pitchfork bifurcation (generically) emerges when the Σ-line crosses (AC), as $\varepsilon_u$ goes through $\varepsilon^{s}_{u}$.13

A flip bifurcation occurs when the Σ-line crosses (AB), as $\varepsilon_u$ goes through $\varepsilon^{f}_{u}$. Finally, a Hopf bifurcation (generically) arises when the Σ-line crosses the segment $[BC]$, as $\varepsilon_u$ goes through $\varepsilon^{h}_{u}$.14

The Σ-line has a slope $S$ given by:

$$S = \frac{\eta \, \varepsilon_u}{c^*(1-\gamma)} - \eta \, \frac{\varepsilon_u}{c^*(1-\gamma)} + \frac{(1-\gamma)(1-\eta_1-\eta_2)(\gamma-\eta_1)}{1-\gamma + \mu^*(\gamma-\eta_1)} + \frac{1-\eta_1-\rho}{1+\alpha} \, \eta^*(\gamma-\eta_1)$$

(36)

We further note that the Σ-line is characterized by the endpoint $(T (+\infty), D (+\infty)) \equiv (\eta_1, 0)$ and the starting point given by:

$$T(0) = \frac{1}{\varepsilon_u} \left\{ \frac{\varepsilon_u}{c^*(1-\gamma)} + \frac{(1-\gamma)}{1-\gamma + \mu^*(\gamma-\eta_1)} \left[ 1 - (\mu^*-1) \eta_1 \left( 2 - \frac{\eta_2}{1-\eta_1} \right) \right] \right. $$

$$\left. + \frac{\mu^*(\gamma-\eta_1)}{1-\gamma + \mu^*(\gamma-\eta_1)} \frac{1-\rho + \eta_1/\gamma}{1+\alpha} \right\}$$

(37)

$$D(0) = \frac{1}{\varepsilon_u} \left[ \frac{\eta_1}{c^*(1-\gamma)} (\varepsilon_u - \varepsilon_u) \right]$$

(38)

with

$$1 - T(0) + D(0) = \frac{1}{\varepsilon_u} \frac{(1-\eta_1)}{c^*(1-\gamma)} (\varepsilon_u^{s} - \varepsilon_u)$$

(39)

and

$$\varepsilon_u^{s} \equiv c^*(1-\gamma) \left\{ \frac{\eta_1}{1-\eta_1} \frac{(1-\gamma)(\mu^*-1)}{1-\gamma + \mu^*(\gamma-\eta_1)} \left( 2 - \frac{\eta_2}{1-\eta_1} \right) \right. $$

$$\left. + \frac{\rho}{1+\alpha} \frac{\mu^*(\gamma-\eta_1)}{1-\gamma + \mu^*(\gamma-\eta_1)} \frac{1}{1-\eta_1} \right\}$$

(40)

5.2 Fluctuations of a bubble

As already underlined, to prove the existence of bubble fluctuations, the monetary policy is assumed to neither depend on inflation forecasts, nor on asset prices, i.e. $\alpha = \rho = 0$. Moreover, to highlight the role played by collaterals on the credit share, we begin by analyzing the case where the credit share is constant ($\eta_1 = \eta_2 = 0$).

5.2.1 Constant credit share

When $\gamma$ is constant ($\eta_1 = \eta_2 = 0$), the trace $T(\varepsilon_v)$ and the determinant $D(\varepsilon_v)$ simplify to:

$$T(\varepsilon_v) = 1 + \frac{1}{1 - \varepsilon_v} \left( \frac{\varepsilon_v}{c(1-\gamma)} + \varepsilon_v \right) \quad \text{and} \quad D(\varepsilon_v) = 0$$

13Indeed, we have (generically) an odd number of steady states (see Section 4).

14The bifurcation values of $\varepsilon_v$, $\varepsilon_u^{s}$, $\varepsilon_u^{f}$ and $\varepsilon_u^{h}$, are given in the Appendix.
As a consequence, the slope of the Σ-line is equal to 0.

Since \( T(\varepsilon_v) \) is strictly increasing with \( \varepsilon_v \) and \( T(0) > 1 \), when \( \varepsilon_v \) varies from 0 to \( +\infty \), \( (T(\varepsilon_v), D(\varepsilon_v)) \) moves on the Σ-line which lies on the horizontal axis, starting from the right hand-side of (AC), crossing ±∞ when \( \varepsilon_v \) goes through 1, then (AB) as \( \varepsilon_v = \varepsilon_f \) and ending at (0, 0) (see Figure 1).

**Proposition 4** Let \( \gamma \) be constant (\( \eta_1 = 0 \) and \( \eta_2 = 0 \)). Under Assumptions 1-6, the following generically holds:

The bubbly steady state is a saddle for \( \varepsilon_v < \varepsilon_f \), undergoes a flip bifurcation for \( \varepsilon_v = \varepsilon_f \) and is a sink for \( \varepsilon_v > \varepsilon_f \).

This proposition shows that persistent fluctuations due to self-fulfilling expectations could emerge around the bubbly steady state. Expectation-driven fluctuations and two-period cycles occur under a significant income effect. This confirms the well-known result obtained by Grandmont (1985, 1986) even if we obtain it in a model with two assets, a bubble and money hold for a transaction.
motive. The economic intuition is the following. If agents expect an increase in inflation from period \( t \) to \( t+1 \), then they reduce their first-period consumption, and increase their global savings, \( b_t + m_t \). As \( \gamma \) is constant, portfolio choice is unchanged. However, agents reduce their second-period consumption, implying a decrease of the return on savings. It follows a raise of inflation and a reduction of the bubble value. Thus, the beliefs are self-fulfilling and expectation-driven fluctuations could emerge.

### 5.2.2 The role of collaterals

We analyze the role of collaterals on expectation-driven fluctuations taking into account that \( \eta_1 \) is not too large. However, we admit either \( \eta_1 < \gamma \), or \( \eta_1 > \gamma \).

In the first case, collaterals have a weak effect on credit, while collaterals have a large effect on credit in the second one.

To identify the indeterminacy properties of the steady state \( b^* = 1 \), we study how the \( \Sigma \)-line evolves in the \((T, D)\) plane in function of \( \eta_1 \). We assume

\[
1 > \frac{\eta_1}{1 - \eta_1} (\mu^* - 1) \left( 2 - \frac{\eta_2}{1 - \eta_1} \right) \tag{41}
\]

This assumption appears reasonable since it is satisfied for \( \eta_1 \) not too large, i.e. a not excessively large effect of collaterals on the credit share.

From direct inspection of the different bifurcation values \( \varepsilon_v', \varepsilon_v^h \) and \( \varepsilon_v^s \), we deduce that:

\[
\varepsilon_v^s < \varepsilon_v^h < \varepsilon_v'
\]

Inequality (42) means that \( T(\varepsilon_v) \) increases with \( \varepsilon_v \). Hence, \((T(\varepsilon_v), D(\varepsilon_v))\) goes rightward crossing \( \pm \infty \) when \( \varepsilon_v \) goes through \( \varepsilon_v^s \).

Let \( S_C > 0 \) be the critical value of \( S \) such that the \( \Sigma \)-line goes through the point \( C \).\(^{16}\) When \( \eta_1 < \gamma \), it can easily be shown from equation (36) and inequality (42) that the \( \Sigma \)-line has a slope in \((0, S_C)\) under Assumption 7. In addition, the \( \Sigma \)-line starts inside the triangle \( ABC \) for \( \varepsilon_v < \varepsilon_u < \varepsilon_u^s \); then crosses \((AC)\) as \( \varepsilon_v = \varepsilon_v^s \), goes inside \( ABC \) by crossing \((AB)\) as \( \varepsilon_v = \varepsilon_u^s \) and ends at \((T(\infty), D(\infty))\) on the horizontal axis.\(^{17}\) When \( \varepsilon_u > \varepsilon_u^s \), the \( \Sigma \)-line starts from outside \( ABC \), on the right of \((AC)\) (see Figure 2).

The \( \Sigma \)-line has still a slope \( S \) in \((0, S_C)\) when \( \eta_1 > \gamma \) and \( \varepsilon_u > \varepsilon_v \), but a slope \( S \) in \((S_B, 0)\) when \( \eta_1 > \gamma \) and \( \varepsilon_u < \varepsilon_v \), where \( S_B < 0 \) is the critical value of \( S \) such that the \( \Sigma \)-line goes through the point \( B \).\(^{18}\) Since \( T(\varepsilon_v) \) increases with \( \varepsilon_v \), the \( \Sigma \)-line points upwards (downwards) to the right when \( \varepsilon_u > \varepsilon_u \) (\( \varepsilon_u < \varepsilon_u \)). Moreover, the \( \Sigma \)-line always starts on the right-hand side of \((AC)\).\(^{19}\)

\(^{15}\)When \( \eta_1 \) is large enough, results are qualitatively similar except that Hopf bifurcations are possible.

\(^{16}\)0 < \( S_C = 1/(2 - \eta_1) \) < 1.

\(^{17}\)When \( \eta_1 < \gamma \), \( 1 - T(0) - D(0) > 0 \) is satisfied for \( \varepsilon_u < \varepsilon_u < \varepsilon_u^s \).

\(^{18}\)-1 < \( S_B = -1/(2 + \eta_1) \) < 0.

\(^{19}\)When \( \eta_1 > \gamma \), \( \varepsilon_u^s < \varepsilon_u < \varepsilon_u \). Hence \( 1 - T(0) - D(0) < 0 \).
the Σ-line goes inside ABC by crossing (AB) as $\varepsilon_v = \varepsilon_v^I$ and ends at $(T (+\infty), D (+\infty))$ on the horizontal axis (see Figure 2).

**Proposition 5** (η1 > 0 collaterals matter)

Under Assumptions 1-7, the following generically holds:

1. If $\eta_1 < \gamma$ and $\varepsilon_u \in (\tilde{\varepsilon}_u, \varepsilon_u^*)$: the steady state is a sink for $\varepsilon_v < \varepsilon_v^*$, undergoes a pitchfork bifurcation for $\varepsilon_v = \varepsilon_v^*$, is a saddle for $\varepsilon_v \in (\varepsilon_v^*, \varepsilon_v^I)$, undergoes a flip bifurcation for $\varepsilon_v = \varepsilon_v^I$, is a sink for $\varepsilon_v > \varepsilon_v^I$.

2. If $\eta_1 < \gamma$ and $\varepsilon_u > \varepsilon_u^*$ or if $\eta_1 > \gamma$: the steady state is a saddle for $\varepsilon_v < \varepsilon_v^I$, undergoes a flip bifurcation for $\varepsilon_v = \varepsilon_v^I$, is a sink for $\varepsilon_v > \varepsilon_v^I$.

Proposition 5 establishes that local indeterminacy and endogenous cycles could occur under sufficiently large income effects ($\varepsilon_v > \varepsilon_v^I$), but also more surprisingly under sufficiently weak income effects ($\varepsilon_u < \varepsilon_u^*$). This requires that collaterals matter, but have a sufficiently weak effect on the credit share ($\eta_1 < \gamma$). Interestingly, this is also associated to the multiplicity of steady
states (see Proposition 3). The intuition is the following. If agents expect an increase in inflation from period $t$ to $t + 1$, then they reduce their first-period consumption and increase their savings if the income effect dominates. The intuition is similar to the previous one with a constant $\gamma$. But, as collaterals now matter, a change in the portfolio structure occurs: an expected inflation rise generates now a portfolio effect, which leads to expectation-driven fluctuations when income effects are weak. Because the return on money is lower than the return on asset, agents reallocate savings from the monetary savings $m_t$ to the bubble $b_t$. Since collaterals matter, households consume less by cash, which implies that the return on money decreases. An effective rise of inflation takes place, i.e. expectations are self-fulfilling.

We have shown that a rational bubble could experience endogenous fluctuations driven by the volatility of agents’ expectations. We turn now to the analysis of the (de-)stabilizing role of monetary policy.

6 The (de-)stabilizing role of monetary policy

The question we address now is: could a monetary policy protect the economy against such fluctuations driven by the volatility of agents’ expectations? The issue has been essentially explored within economies without collaterals and portfolio choice (Grandmont (1985, 1986), Sorger (2005)). While Grandmont (1985, 1986) describes as stabilizing a monetary policy which is able to coordinate expectations, a monetary policy is stabilizing in our framework as soon as it could reduce the range of parameter values for which expectation-driven fluctuations occur. In line with Sorger (2005), we study first an inflation forecast targeting rule which responds only to the expected inflation ($\alpha \neq 0$ and $\rho = 0$). Note that such a rule is of interest since it seems to be more relevant to describe the US monetary policy than a rule based on observed inflation (Bernanke (2010)). Second, we analyze a rule taking into account asset prices. Considering such a monetary rule may be appropriate in an economy in which a bubble could persist and experience fluctuations. Thus, we contribute to the debate initiated by Bernanke and Gertler (1999, 2001), whether central banks should react to movements in asset prices. We extend the inflation targeting rule, by including directly asset price level ($\alpha \neq 0$ and $\rho \neq 0$). Our findings mitigate the conclusions issued by Bernanke and Gertler (1999, 2001), Gilchrist and Leahy (2002), and more recently Carlstrom and Fuerst (2007). Compared with the first two ones, we show that such a rule may have a significant effect on the macroeconomic stability. Unlike to Calstrom and Fuerst (2007), such a rule stabilizes fluctuations.

To do this, we examine how local dynamics are altered by the implementation of monetary policy. As previously, our analysis focuses on a not too large value of $\eta_1$. 

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6.1 Inflation forecast targeting

Following Sorger (2005), we start by analyzing the stabilizing role of inflation forecast targeting on the emergence of fluctuations. However, we consider a model with two assets, a bubble and money hold for a transaction motive, and most importantly in which collaterals matter. We will see that this last feature may reverse the conclusions obtained by Sorger (2005) that an active rule on inflation forecasts tends to destabilize.

Hence, the monetary authority conducts a monetary rule given by (14) with $\rho = 0$:

$$\mu_t = \mu^*(\frac{\pi_t + 1}{\mu^*})^{-\alpha}, \quad \text{with } \alpha \geq -1.$$  

The purpose is to evaluate if inflation forecast targeting stabilizes or rather destabilizes expectation-driven fluctuations around the bubbly steady state. As in the previous section, focusing on the geometrical elements, we can easily check that under Assumption 7, the slope $S$ is always positive when $\eta_1 < \gamma$ and could be negative when $\eta_1 > \gamma$.

For any value of $\alpha$, when $\eta_1 < \gamma$, the $\Sigma$-line has a slope between $(0, \infty)$ and starts inside the triangle $ABC$ for $\varepsilon_u < \varepsilon_u < \varepsilon_u^*$, then crosses $(AC)$ as $\varepsilon_v = \varepsilon_v^*$, goes inside $ABC$ by crossing $(AB)$ as $\varepsilon_v = \varepsilon_v^*$ and ends at $(T(+\infty), D(+\infty))$ on the horizontal axis.

When $\eta_1 > \gamma$, $\bar{\varepsilon}_v \equiv \frac{1 - \gamma + \mu^*(\gamma - \eta_1)}{1 - \gamma + \mu^*(\gamma - \eta_1)}$ could be negative for some values of $\alpha \in (-1, 0)$. To keep things as simple as possible, we restrict our analysis to the cases where $\bar{\varepsilon}_v > 0$. We assume:

**Assumption 8**

$$\alpha > -\frac{1 - \gamma + \mu^*(\gamma - \eta_1)}{1 - \gamma} \equiv \tilde{\alpha}$$

Note that $\tilde{\alpha} < 0$ may be close to $-1$, especially if the difference between $\eta_1$ and $\gamma$ is not too large or under low distortions ($\eta_1$ and $\gamma$ close to zero). Hence, this assumption does not seem to be too restrictive. For any value of $\alpha > \tilde{\alpha}$, the $\Sigma$-line has a slope $S$ between $(0, S_C)$ (between $(S_B, 0)$) when $\varepsilon_u \geq \varepsilon_u$ ($\varepsilon_u < \varepsilon_u$) and always starts on the right-hand side of $(AC)$. Afterwards, the $\Sigma$-line, pointing upwards (downwards) when $\varepsilon_u \geq \varepsilon_u$ ($\varepsilon_u < \varepsilon_u$), goes inside $ABC$ by crossing $(AB)$ below $B$ as $\varepsilon_v = \varepsilon_v^A$ and ends at $(T(+\infty), D(+\infty))$.

To summarize, when $\eta_1 < \gamma$, local indeterminacy occurs if $\varepsilon_u < \varepsilon_u^*$ or if $\varepsilon_v > \varepsilon_v^*$ and, when $\eta_1 > \gamma$, if $\varepsilon_v > \varepsilon_v^*$. What is important is to know how these critical bifurcation values, $\varepsilon_v^*$ and $\varepsilon_v^*$, vary in function of $\alpha$. Knowing these variations allows to give a picture of the role of the monetary policy on local indeterminacy. The following lemma provides these variations.

**Lemma 4** Under Assumptions 1-7, the effects of the monetary policy on the bifurcation values are given by the following derivatives:
Figure 3: Inflation forecast targeting when $\eta_1 < \gamma$

1. When $\eta_1 < \gamma$, $\frac{d\varepsilon_v^f}{d\alpha} = 0$ and $\frac{d\varepsilon_v^s}{d\alpha} < 0$.

2. When $\eta_1 > \gamma$, $\frac{d\varepsilon_v^f}{d\alpha} > 0$.

Proof. See the Appendix.

If $\eta_1 < \gamma$, the higher $\alpha$ is, the lower $\varepsilon_v^f$ is. On the other hand, $\varepsilon_v^s$ does not depend on $\alpha$ (see Figure 3). This allows us to deduce the (de-)stabilizing role of an inflation targeting monetary policy.

**Proposition 6** Under Assumptions 1-7, the following generically holds when $\eta_1 < \gamma$:

1. The more active the monetary policy is ($\alpha$ higher), the more destabilizing it is in the neighborhood of the steady state $b^*$, for high income effects.

2. A more active monetary policy has no destabilizing or stabilizing effects in the neighborhood of the steady state $b^*$, for weak income effects.

This proposition suggests that when $\varepsilon_v > \varepsilon_v^f$, an active inflation forecast targeting promotes the emergence of expectation-driven fluctuations, while a passive policy may rule out it. This is in accordance with Sorger (2005), even if unlike him we assume that the monetary transfer is distributed at the first period of life and there is a portfolio choice. However, such a monetary policy...
does not (de-)stabilize fluctuations which occur for low $\varepsilon_v$. This also means that it has no impact on the multiplicity of steady states.

Focusing now on the case $\eta_1 > \gamma$, we recall that indeterminacy requires $\varepsilon_v > \varepsilon_f$. From Lemma 4, we deduce that the higher $\alpha$ is, the larger is $\varepsilon_f$. The next proposition summarizes the role of an inflation targeting monetary policy when $\eta_1 > \gamma$:

![Figure 4: Inflation forecast targeting when $\eta_1 > \gamma$](image)

**Proposition 7** Under Assumptions 1-8, the following generically holds when $\eta_1 > \gamma$: the more active the monetary policy is, the more stabilizing it is in the neighborhood of the steady state $b^*$, for high income effects.

This proposition means that, for high income effects, a sufficiently active monetary policy hardens the condition on $\varepsilon_v$, which guarantees local indeterminacy. An active inflation forecast targeting could stabilize the fluctuations of the rational bubble. This mitigates the clear-cut conclusion pronounced by Sorger (2005), i.e. if one focuses on stabilization, an active inflation targeting is preferable to an active inflation forecast targeting.

To summarize, the stabilizing role of an active inflation forecast targeting monetary policy depends on the sensibility of the credit share with respect to collaterals. A sufficiently large sensitivity of credit share to collaterals is the source of the stabilizing role of an active monetary policy, because it reinforces the role of the portfolio effects. However, such a policy rule has no effect on fluctuations that occur for low income effects.
6.2 Monetary rule with asset price level

To answer the question addressed in particular by Bernanke and Gertler (2001) “Should central banks respond to movements in asset prices?”, we enrich the inflation forecast targeting rule. The money growth rate responds also to the asset price deviation from the stationary value $b^*$:

$$\mu_t = \mu\left(\frac{\pi_t+1}{\mu^*}\right)^{-\alpha} \left(\frac{b_t}{b^*}\right)^{-\rho}, \quad \text{with } \alpha \geq -1, \rho \geq -1.$$

We examine if taking into account asset prices in the monetary rule may significantly alter the (de-)stabilizing role of monetary policy. Thus, we extend our previous analysis to changes in asset prices for a given value of $\alpha$.

When $\rho$ increases from $-1$ to $+\infty$, the location of the $\Sigma$-line in the plane $(T(\varepsilon_v), D(\varepsilon_v))$ differs from the previous analysis.\(^\text{21}\) From equation (36), we can easily show that the slope $S$ of the $\Sigma$-line is an increasing function of $\rho$ when $\eta_1 < \gamma$ or when $\eta_1 > \gamma$ and $\varepsilon_u < \bar{\varepsilon}_u$, and a decreasing function of $\rho$ when $\eta_1 > \gamma$ and $\varepsilon_u \geq \bar{\varepsilon}_u$.

When $\eta_1 < \gamma$, the $\Sigma$-line makes a counterclockwise rotation around the endpoint $(T(+\infty), D(+\infty))$ when $\rho$ increases, starting with a slope $S$ between $(0, S_C)$. More precisely, we can easily show from equation (36) that for $\rho = -1$, the $\Sigma$-line has a positive slope, lower than $S_C$. Afterwards, when $\rho$ increases, it goes first through $C$, second through $A$, then $B$, and becomes flat, i.e. close to zero. Moreover, the $\Sigma$-line can start inside the triangle $ABC$ for low $\varepsilon_v$.

![Figure 5: Monetary policy with asset price level when $\eta_1 < \gamma$](image)

\(^{21}\)Obviously, for $\rho = 0$, Propositions 6 and 7 apply.
When \( \eta_1 > \gamma \) and \( \varepsilon_u \geq \bar{\varepsilon}_u \), the \( \Sigma \)-line makes a clockwise rotation around the endpoint \((T(+\infty), D(+\infty))\) when \( \rho \) increases, pointing upwards. For \( \rho = -1 \), the slope \( S \) could be between \((S_B, 0)\) (see equation (36)). Therefore, the \( \Sigma \)-line goes through the point \( B \), then the point \( A \) and \( C \) to become flat when \( \rho \) increases. As before, the \( \Sigma \)-line can start inside the triangle \( ABC \) for low \( \varepsilon_v \).

When \( \varepsilon_u < \bar{\varepsilon}_u \), the \( \Sigma \)-line makes a counterclockwise rotation around the endpoint \((T(+\infty), D(+\infty))\) when \( \rho \) increases, pointing downwards. For \( \rho = -1 \), the slope \( S \) could be between \((0, S_C)\) (see equation (36)). Hence, the \( \Sigma \)-line goes through the point \( C \), then the point \( A \) and \( B \) to become flat when \( \rho \) increases. The \( \Sigma \)-line can also start inside the triangle \( ABC \) for low \( \varepsilon_v \).

To summarize, when \( \eta_1 < \gamma \), local indeterminacy occurs if \( \varepsilon_v < \min(\varepsilon_s, \varepsilon_f, \varepsilon_h) \) and if \( \varepsilon_v > \max(\varepsilon_f, \varepsilon_s, \varepsilon_h) \) (see Figure 5). When \( \eta_1 > \gamma \) and \( \varepsilon_u > \bar{\varepsilon}_u \), indeterminacy arises if \( \varepsilon_v < \min(\varepsilon_s, \varepsilon_f, \varepsilon_h) \) and if \( \varepsilon_v > \max(\varepsilon_f, \varepsilon_s, \varepsilon_h) \) (see Figure 6). Finally, when \( \eta_1 > \gamma \) and \( \varepsilon_u < \bar{\varepsilon}_u \), indeterminacy emerges if \( \varepsilon_v < \min(\varepsilon_s, \varepsilon_f, \varepsilon_h) \) and if \( \varepsilon_v > \max(\varepsilon_f, \varepsilon_s, \varepsilon_h) \) (see Figure 7).

Now, we study how these critical bifurcation values, \( \varepsilon_s, \varepsilon_f \) and \( \varepsilon_h \), vary in function of \( \rho \).

**Lemma 5** Under Assumptions 1-7, the effects of the monetary policy \( \rho \) on the bifurcation values are given by the following derivatives:

1. When \( \eta_1 < \gamma \), \( \frac{d\varepsilon_s}{d\rho} > 0 \), \( \frac{d\varepsilon_f}{d\rho} < 0 \) and \( \frac{d\varepsilon_h}{d\rho} = 0 \).
2. When \( \eta_1 > \gamma \), \( \frac{d\varepsilon_s}{d\rho} < 0 \), \( \frac{d\varepsilon_f}{d\rho} > 0 \) and \( \frac{d\varepsilon_h}{d\rho} = 0 \).

**Proof.** See the Appendix.

Let \( \rho^{fs} \) be the crossing point between \( \varepsilon_s^* \) and \( \varepsilon_f^* \). The next proposition summarizes the role of a monetary policy on local indeterminacy for a given value of \( \alpha \):

**Proposition 8** Under Assumptions 1-8, the following generically holds:

1. If \( -1 < \rho < \rho^{fs} \), the more passive the monetary policy is, the more stabilizing it is in the neighborhood of the steady state \( b^* \).
2. If \( \rho^{fs} < \rho \), the more active the monetary policy is, the more stabilizing it is in the neighborhood of the steady state \( b^* \).

Whatever the role of collaterals and the degree of income effect, a sufficiently active monetary policy taking asset prices into account, i.e. \( \rho > \rho^{fs} \), hardens the conditions on \( \varepsilon_v \), which lead to local indeterminacy. An active monetary rule taking asset prices into account makes the economy more stable around the steady state targeted.

---

22 Figures 5-7 give just qualitative illustrations.
23 More details are available upon request.
24 The value of \( \rho^{fs} \) is given in the Appendix. We can easily show that when \( \eta_1 < \gamma, \rho^{fs} > 0 \).
Figure 6: Monetary policy with asset price level when $\eta_1 > \gamma$ and $\varepsilon_u > \bar{\varepsilon}_u$

Recall that multiplicity is connected to $\varepsilon_v < \varepsilon_s$. The next proposition summarizes, for a given value of $\alpha$, the role of a monetary policy on multiplicity of steady states, which can be a source of global indeterminacy:

**Proposition 9** Under Assumptions 1-8, the following generically holds:

1. When $\eta_1 < \gamma$, the more passive the monetary policy is, the weaker the range of values $\varepsilon_v$ under which the multiplicity of steady states exists is.

2. When $\eta_1 > \gamma$, the more active the monetary policy is, the weaker the range of values $\varepsilon_v$ under which the multiplicity of steady states exists is.

We deduce from this proposition that, when $\eta_1 < \gamma$, a passive monetary rule makes global indeterminacy less likely to emerge. This means that even if an active monetary rule can locally stabilize, a passive one may stabilize both locally and globally. In other words, the steady state $b^*$ is getting determinate, and multiplicity of steady states disappears. In any case, once asset prices are integrated, a monetary rule allows to stabilize for weak income effects and has an impact on the multiplicity of steady states, which was not the case without taking asset prices into account.

When $\eta_1 > \gamma$, we have an opposite result: a sufficiently active monetary policy, taking asset prices into account, makes global indeterminacy less likely to emerge. The steady state $b^*$ becomes determinate and multiplicity of steady solutions not too far from $b^* = 1$. This is what we call a globally stabilizing monetary policy.

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25 As it is clear from the steady state analysis (see Section 4.2), $\varepsilon_v < \varepsilon_s$ is sufficient condition to get multiplicity. Therefore, ensuring $\varepsilon_v > \varepsilon_s$ is a way to eliminate, at least, stationary solutions not too far from $b^* = 1$. This is what we call a globally stabilizing monetary policy.
states disappears. This means that when collaterals matter sufficiently, an active monetary rule can stabilize both locally and globally.

Hence, a monetary rule including asset prices could locally and globally stabilize fluctuations unlike a monetary rule which responds only to inflation. More precisely, when collaterals have weak effects on the credit share, a passive monetary rule should be led, and when collaterals matter sufficiently, an active one should be led. Following the debate initiated by Bernanke and Gertler (1999, 2001) and recently advanced by several Governors of central bank or practitioners of monetary policy (Yellen (2009), Bernanke (2010, 2011)), we provide a clear-cut conclusion. Under credit market imperfections, a monetary policy should respond to movements in asset prices to stabilize expectation-driven fluctuations.

7 Concluding remarks

We present a monetary overlapping generations model with two assets, bubble and money needed to transaction motives, and effects of collaterals. The endogenous portfolio choice promotes expectation-driven fluctuations and the multiplicity of steady states for low income effects and arbitrarily weak distortions.

When monetary policy only depends on expected inflation, a sufficiently active policy can stabilize expectation-driven fluctuations occurring for high income effects, if the effect of collaterals on the credit share is large enough. The role of collaterals in our model may overturn the conclusions obtained
by Sorger (2005). However, such a rule has no impact on the occurrence of expectation-driven fluctuations for low income effects and on multiplicity of steady states.

In our model, a monetary policy including asset prices may stabilize aggregate fluctuations. Indeed, a sufficiently active rule stabilizes fluctuations around the steady state both for low and high income effects, whatever the sensitivity of the credit share to collaterals. In addition, a monetary rule on asset prices can locally and globally stabilize fluctuations unlike a monetary rule which only responds to inflation. More precisely, when collaterals have weak effects on the credit share, a passive monetary rule should be led, and when collaterals matter sufficiently, an active one. Our results mitigate the conclusions pronounced by Bernanke and Gertler (1999, 2001), and reinforced recently for instance by Carlstrom and Fuerst (2007). To reply to the question raised by Bernanke and Gertler (2001), central banks that take care of macroeconomic instability should respond to movements in asset prices.

Note that this previous analysis is also conducted for a given value of $\alpha$ (see Section 6.2). However, our results are not qualitatively modified by $\alpha$. Indeed, for a sufficiently large effect of collaterals on the credit share ($\gamma < \eta_1$), an increase of $\alpha$ reinforces mechanisms which lead to stabilize both locally and globally, taking into account that $\rho$ is not too weak. Nonetheless, for a weak effect of collaterals ($\gamma > \eta_1$), an increase of $\alpha$ reinforces mechanisms which locally destabilize for high $\epsilon_v$ or a sufficiently large $\rho$.

Finally, our qualitative results are obtained for all value of the monetary growth rate’s target $\mu^*$. Therefore, these conclusions are compatible with a target $\mu^*$ which maximizes households’ welfare at the steady state. When the monetary rule including asset prices is not too passive, $\mu^* = 1$ is an optimal rule. We recognize the Friedman rule: no money growth and a nominal interest rate equals to zero ($\mu^* = 1 + i = 1$). The intertemporal choices of households are no longer affected by the distortions due to money holdings. From equation (29) at the normalized steady state, we obtain $u' (e - 1/[1 - \gamma (1)]) = \beta^* v' (1/[1 - \gamma (1)])$, which corresponds to the Phelps Golden rule in a monetary overlapping generations model without population growth. This means that a stabilizing monetary rule, which eliminates the cost of fluctuations, may be in accordance with welfare maximisation at the steady state.

---

26 This could be shown using the bifurcation values $\epsilon_f^*, \epsilon_h^*$ and $\epsilon_v^*$ (see the Appendix).
8 Appendix

Proof of Lemma 1

We maximize the Lagrangian function:

\[
\begin{align*}
&u(c_t) + \beta v(d_{t+1}) \\
&+ \lambda_{1t} \left( c_t + \pi_t - \pi_t m_{t+1} - b_t - c_t \right) \\
&+ \lambda_{2t} \left( m_{t+1} + \frac{1}{\pi_t} b_t - d_{t+1} \right) \\
&+ \lambda_{3t} \left( m_{t+1} - \gamma(b_t) d_{t+1} \right)
\end{align*}
\] (43)

with respect to \((c_t, d_{t+1}, m_{t+1}, b_t, \lambda_{1t}, \lambda_{2t}, \lambda_{3t})\). Since \(\lambda_{1t} = u'(c_t) > 0\), then (7) becomes binding. Because \(\lambda_{2t} = \lambda_{1t} + \gamma'(b_t) d_{t+1} \) (44)

\[
\lambda_{3t} = \lambda_{1t} \left( \pi_{t+1} - \frac{1 + \pi_{t+1} \gamma'(b_t) d_{t+1}}{\pi_{t+1}} \right)
\] (45)

strict positivity of \(\lambda_{2t}\) and \(\lambda_{3t}\) requires

\[
\pi_{t+1} > \frac{1 + \gamma'(b_t) \pi_{t+1} d_{t+1}}{\pi_{t+1} + \gamma'(b_t) d_{t+1}} > 0
\] (46)

Inequality \(1 + \gamma'(b_t) \pi_{t+1} d_{t+1} > 0\) is equivalent to (10). Moreover, \(i_{t+1} > 0\) implies \((1 + i_{t+1}) + \gamma'(b_t) \pi_{t+1} d_{t+1} > 1 + \gamma'(b_t) \pi_{t+1} d_{t+1}\), which ensures that (46) holds. ■

Proof of Lemma 2

We compute the Hessian matrix of the Lagrangian function (43) with respect to \((c_t, d_{t+1}, m_{t+1}, b_t, \lambda_{1t}, \lambda_{2t}, \lambda_{3t})\):

\[
H \equiv \begin{bmatrix}
  u'' & 0 & 0 & 0 & -1 & 0 & 0 \\
 0 & \beta v'' & 0 & -\lambda_3 \gamma' & 0 & -1 & -\gamma \\
 0 & 0 & 0 & 0 & -\pi & 1 & 1 \\
 0 & -\gamma \lambda_3 & 0 & -\gamma^2 \lambda_3 d & -1 & \frac{1+i}{\pi} & -\gamma d \\
-1 & 0 & -\pi & -1 & 0 & 0 & 0 \\
0 & -1 & 1 & \frac{i}{\pi} & 0 & 0 & 0 \\
0 & -\gamma & 1 & -\gamma d & 0 & 0 & 0
\end{bmatrix}
\] (47)

In order to get a strict local maximum, we need to check the negative definiteness of \(H\) over the set of points satisfying the constraints. Let \(p\) and \(n\) the numbers of constraints and variables. If the determinant of \(H\) has sign \((-1)^n\)

\[\text{For simplicity, the arguments of the functions and the time subscripts are omitted.}\]
and the last $n - p$ diagonal principal minors have alternating signs, then the optimum is a regular local maximum. In our case $n = 4$ and $p = 3$. Therefore, we simply require $\det H > 0$, that is,

$$
\det H = -u'' \left[ \pi \left( \gamma' d + \frac{1 + i}{\pi} \gamma \right) - (\gamma - 1) \right]^2 \\
- \beta v'' \left( \gamma' d + \frac{1 + i}{\pi} \right)^2 \\
+ \lambda_3 (1 - \gamma) \left[ 2 \left( \gamma'^2 d + \gamma' \frac{1 + i}{\pi} \right) + \gamma'' d (1 - \gamma) \right] > 0 \quad (48)
$$

Using (8) and (9), we find $d_{t+1} = \frac{b_t}{1 - \gamma(b_t)} \frac{1 + i}{\pi_t + 1}$. Substituting in (48), we get:

$$
\det H = - (\frac{1 + i}{\pi})^2 \left[ \chi_0 + \chi_1 u'' + \beta v'' (1 - \eta_1)^2 \right] > 0 \quad (49)
$$

where

$$
\chi_0 \equiv \lambda_3 \eta_1 \frac{1 - \gamma}{b} \left[ 2 \left( 1 - \eta_1 \right) - \eta_2 \right] \frac{1 - \gamma}{1 + i} \\
\chi_1 \equiv \pi (\gamma - \eta_1) + \frac{1 - \gamma}{1 + i}
$$

The second order condition is satisfied if

$$
\varepsilon_u > \frac{c(1 - \gamma)^2}{[1 - \gamma + (1 + i) (\gamma - \eta_1)]^2 \eta_1 (2 - \frac{\eta_2}{1 - \eta_1}) \equiv \bar{\varepsilon}_u} \quad (50)
$$

**Proof of Proposition 1**

A steady state $b$ is a solution of $g(b) = h(b)$, with:

$$
g \left( b \right) \equiv \frac{u' \left( e - \frac{b}{1 - \gamma(b)} \right)}{\beta v' \left( \frac{b}{1 - \gamma(b)} \right)} \quad (51)
$$

$$
h \left( b \right) \equiv \frac{1 - \eta_1 (b)}{1 - \gamma (b) + \gamma (b) - \eta_1 (b) \mu (b)} \quad (52)
$$

Since $\frac{b}{1 - \gamma(b)}$ is increasing in $b$ and $c > 0$, then $b < \bar{b}$, where $\bar{b} > 0$ is such that $e = \frac{\bar{b}}{1 - \gamma(\bar{b})}$. Therefore, all the stationary solutions $b$ belong to $(0, \bar{b})$. 

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28 For simplicity, the arguments of the functions and the time subscripts are omitted.

29 Under Assumption 3, $b$ is well defined.
To prove the existence of a stationary solution \( b \), we use the continuity of \( g(\bar{b}) \) and \( h(\bar{b}) \). From (31), (51) and (52), we determine the boundary values of \( g(\bar{b}) \) and \( h(\bar{b}) \):

\[
\lim_{\bar{b} \to b} g(\bar{b}) = \frac{u'(c)}{\beta u'(0)} = 0^+, \quad \lim_{\bar{b} \to b} g(\bar{b}) = +\infty
\]

\[
\lim_{\bar{b} \to 0} h(\bar{b}) = \frac{1 - \eta_1(0)}{1 - \gamma(0) + [\gamma(0) - \eta_1(0)] \mu(0)} \geq 0
\]

\[
\lim_{\bar{b} \to b} h(\bar{b}) = \frac{1 - \eta_1(\bar{b})}{1 - \gamma(\bar{b}) + [\gamma(\bar{b}) - \eta_1(\bar{b})] \mu(\bar{b})} > 0
\]

We have \( \lim_{\bar{b} \to \bar{b}} g(\bar{b}) > \lim_{\bar{b} \to \bar{b}} h(\bar{b}) \). If \(-1 < \rho \leq 0, \mu(0) < +\infty\), which implies \( \lim_{\bar{b} \to b} g(\bar{b}) < \lim_{\bar{b} \to b} h(\bar{b}) \). However, if \( \rho > 0, \mu(0) = +\infty \), which implies \( g(0) = h(0) \). In this case, the existence of a steady state \( b \in (0, \bar{b}) \) solving \( g(\bar{b}) = h(\bar{b}) \), is ensured by \( \varepsilon_g(\bar{b}) < \varepsilon_h(\bar{b}) \), where:

\[
\varepsilon_g(\bar{b}) = \varepsilon_u(c) \frac{1 - \eta_1(\bar{b})} {1 - \gamma(\bar{b})} \frac{b} {c} + [1 - \eta_1(\bar{b})] \varepsilon_v(d) > 0
\]

\[
\varepsilon_h(\bar{b}) = \frac{h'(\bar{b}) b}{h(\bar{b})} = \eta_1(\bar{b}) \frac{[1 - \gamma(\bar{b})] [\mu(\bar{b}) - 1] [2 [1 - \eta_1(\bar{b})] - \eta_2(\bar{b})]} {[1 - \eta_1(\bar{b})] [1 - \gamma(\bar{b}) + \mu(\bar{b}) [\gamma(\bar{b}) - \eta_1(\bar{b})]] + \rho \frac{\mu(\bar{b}) [\gamma(\bar{b}) - \eta_1(\bar{b})]} {1 + \alpha 1 - \gamma(\bar{b}) + \mu(\bar{b}) [\gamma(\bar{b}) - \eta_1(\bar{b})]}]
\]

Under Assumption 4, this last condition is satisfied. A sufficient condition for uniqueness is \( \varepsilon_h(\bar{b}) < \varepsilon_g(\bar{b}) \) for all \( b \in (0, \bar{b}) \), or equivalently,

\[
\varepsilon_u(c) \frac{b} {c(1 - \gamma(\bar{b}))} + \varepsilon_v(d) > \frac{\eta_1(\bar{b})} {1 - \eta_1(\bar{b})} \frac{[1 - \gamma(\bar{b})] [\mu(\bar{b}) - 1]} {1 - \gamma(\bar{b}) + \mu(\bar{b}) [\gamma(\bar{b}) - \eta_1(\bar{b})]}
\]

\[
\left( \frac{2 - \eta_2(\bar{b})} {1 - \eta_1(\bar{b})} \right) + \frac{\rho} {1 + \alpha 1 - \gamma(\bar{b}) + \mu(\bar{b}) [\gamma(\bar{b}) - \eta_1(\bar{b})]} \frac{\mu(\bar{b}) [\gamma(\bar{b}) - \eta_1(\bar{b})]} {1 - \eta_1(\bar{b})}
\]

\textbf{Proof of Proposition 3}

Using the notations of the proof of Proposition 1, we know that \( g(\bar{b}) > h(\bar{b}) \) and under Assumption 4, \( g(\bar{b}) < h(\bar{b}) \) for \( b > 0 \) but arbitrarily close to 0. Since \( b^* = 1 \) is a steady state, we have: \( g(1) = h(1) \). If inequality (32) is satisfied, we have \( g'(1) < h'(1) \), then by continuity there exist at least two other steady states, \( b_1 \) and \( b_2 \) such that \( b_1 < 1 < b_2 \). The number of steady states is generically odd. ■
Proof of Lemma 3

Substituting (27) in (25) and differentiating around the NSS with respect to $b_{t-1}, b_t$ and $b_{t+1}$, we obtain the following linearization:

\[
\left[\varepsilon_v - \frac{1 - \gamma + \mu^* (\gamma - \eta_1)}{1 - \gamma + \mu^* (\gamma - \eta_1)} \frac{1}{1 + \alpha}\right] db_{t+1} + \left[\frac{1}{(1 - \gamma) c^*} - \eta_1 \varepsilon_v + 1\right. \\
+ \frac{(\gamma - \eta_1) \mu^*}{1 - \gamma + \mu^* (\gamma - \eta_1)} \frac{1 - \rho + \eta_1}{1 + \alpha} (1 - \gamma) (\mu^* - 1) \eta_1 \left(2 - \frac{\eta_2}{1 - \eta_1}\right) \right] db_t \\
- \left[\frac{\eta_1}{1 - \gamma} + \frac{\mu^* (\gamma - \eta_1)}{1 - \gamma + \mu^* (\gamma - \eta_1)} \frac{1}{1 + \alpha}\right] db_{t-1} = 0
\]

Lemma 3 follows.

Bifurcation values of $\varepsilon_v$

$\varepsilon_v^s$ is defined by $1 - T(\varepsilon_v) + D(\varepsilon_v) = 0$:

\[
\varepsilon_v^s = \frac{1}{c^* (1 - \gamma)} (\varepsilon_u^s - \varepsilon_u),
\]

where $\varepsilon_u^s$ is given by (40).

$\varepsilon_v^f$ is defined by $1 + T(\varepsilon_v) + D(\varepsilon_v) = 0$:

\[
\varepsilon_v^f = \frac{\varepsilon_u}{c^* (1 - \gamma)} + \frac{1 - \gamma}{1 - \gamma + \mu^* (\gamma - \eta_1)} \left[\frac{2}{1 + \eta_1} - \frac{\eta_1}{1 + \eta_1} (\mu^* - 1) \left(2 - \frac{\eta_2}{1 - \eta_1}\right)\right] \\
+ \frac{2}{1 + \eta_1} \frac{\mu^* (\gamma - \eta_1)}{1 - \gamma + \mu^* (\gamma - \eta_1)} \frac{1 + \eta_1 / \gamma - \rho / 2}{1 + \alpha}
\]

$\varepsilon_v^h$ is defined by $D = 1$:

\[
\varepsilon_v^h = \frac{1 - \gamma}{1 - \gamma + \mu^* (\gamma - \eta_1)} + \frac{\mu^* (\gamma - \eta_1)}{1 - \gamma + \mu^* (\gamma - \eta_1)} \frac{1}{1 + \alpha} - \frac{\eta_1}{c^* (1 - \gamma)} \varepsilon_u
\]

Proof of Lemma 4

\[
\frac{d\varepsilon_v^s}{d\alpha} = -\frac{1}{(1 + \alpha)^2} \frac{\mu^* (\gamma - \eta_1)}{1 - \gamma + \mu^* (\gamma - \eta_1)} \frac{\rho}{1 - \eta_1}
\]

\[
\frac{d\varepsilon_v^f}{d\alpha} = -\frac{1}{(1 + \alpha)^2} \frac{2}{1 + \eta_1} \frac{\mu^* (\gamma - \eta_1)}{1 - \gamma + \mu^* (\gamma - \eta_1)} (1 + \eta_1 / \gamma - \rho / 2)
\]
Lemma 4 follows.

**Proof of Lemma 5**

\[
\frac{d\varepsilon^s}{d\rho} = \frac{1}{1 + \alpha} \frac{\mu^* (\gamma - \eta_1)}{1 - \gamma + \mu^* (\gamma - \eta_1)} \frac{1}{1 - \eta_1
\]

\[
\frac{d\varepsilon^f}{d\rho} = -\frac{1}{1 + \alpha} \frac{1}{1 + \eta_1} \frac{\mu^* (\gamma - \eta_1)}{1 - \gamma + \mu^* (\gamma - \eta_1)}
\]

\[
\frac{d\varepsilon^h}{d\rho} = 0
\]

Lemma 5 follows.

**Critical value of \(\rho\)**

\(\rho^{fs}\) is defined by \(\varepsilon^f = \varepsilon^s\):

\[
\rho^{fs} = (1 + \alpha) \left\{ \varepsilon_u \frac{1 - \gamma + \mu^* (\gamma - \eta_1)}{\mu^* (\gamma - \eta_1)} (1 - \eta_2^2) + \frac{1 - \gamma}{\mu^* (\gamma - \eta_1)} \left[ 1 - \eta_1 - \eta_1 (\mu^* - 1) \left( 2 - \frac{\eta_2}{1 - \eta_1} \right) \right] + \frac{(1 - \eta_1)(1 + \eta_1/\gamma)}{1 + \alpha} \right\}
\]

**References**


