Fiscal policy and growth through endogenous cycles

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Abstract

We explore the role of fiscal policy in the Matsuyama model (1999, Econometrica) of growth through cycles alternating perpetually between two phases featuring neoclassical investment and neo-Schumpeterian innovation respectively. Subsidies to R&D investment or to the purchase of newly invented intermediate goods can arbitrarily reduce the threshold level of capital per type of intermediate good, beyond which the economy moves from the investment phase to the innovation phase. More importantly, such subsidies can mitigate and eventually eliminate the cycles for significant welfare gains that can be equivalent to as much as 15% rises in consumption at all times.

Keywords: Subsidization; Innovation; Capital accumulation; Cycles; Growth

JEL Classifications: E3, E6, O4

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1. Introduction

One theme of macroeconomics is to explore whether and how government policies can promote output growth or mitigate output fluctuations for welfare gains. However, different macroeconomic models have different policy implications. The neoclassical growth model, pioneered by Solow (1956) and Swan (1956), captures how capital accumulation contributes to growth and why growth eventually halts at a stable steady-state output level per capita under diminishing returns to investment. It may need no role of government intervention so long as consumers choose their consumption path optimally as in Cass (1965) and Koopmans (1965) according to the Welfare Theorems. This policy implication can remain valid even when exogenous shocks are introduced into the neoclassical growth model for the creation of cycles as in the real business cycle models led by Kydland and Prescott (1982) and Long and Plosser (1983).

The neo-Schumpeterian models in the last two decades generate sustainable growth through costly innovations that create new varieties of intermediate goods or improve the quality of existing intermediate goods (e.g. Romer, 1990; Aghion and Howitt, 1992). In these R&D endogenous growth models, R&D activities intended for a new variety or a quality improvement incur a fixed cost, whereas the production of each intermediate good incurs a constant marginal cost; and both the innovation and the intermediate goods production use current final output. Monopoly rights are granted to innovators in order to allow them to recoup their innovational costs. The consequent monopoly pricing reduces the demand for new intermediate goods, causing lower final output and slower growth than their socially optimal levels (static and dynamic losses in efficiency, respectively). The
efficiency losses of monopoly pricing justify government subsidization either to R&D investment or to the purchase of newly created intermediate goods, as shown in Barro and Sala-i-Martin (1995) and Zeng and Zhang (2007) among others. In contrast to the neoclassical growth and the real business cycle models, however, the economy is always on the balanced growth path in such neo-Schumpeterian growth models that are known as the AK model in essence.

Matsuyama (1996, 1999) unifies the neoclassical and the neo-Schumpeterian growth models by assuming that both R&D activities and intermediate goods production can only use available capital from previous savings. Under this neoclassical-style assumption, innovation can break even to recover the fixed cost if and only if capital per type of intermediate good exceeds a critical level for a profitable scale of the demand for newly invented intermediates. Once initial capital exceeds this level and induces innovation, however, part of the initial capital must be used for the fixed innovation cost and the remaining amount of capital for manufacturing intermediate goods declines. Consequently, current innovation, if responding far more sensitively than capital investment to initial abundance in capital per variety, can reduce capital per variety so much that future innovation becomes unprofitable until enough capital is formed again through a neoclassical investment phase. It argues that this is an empirically plausible scenario: The balanced growth path with innovation is unstable and therefore the economy may fluctuate between a Solow investment phase and a Romer innovation phase perpetually, causing cyclical movements in consumption. In a different neo-Schumpeterian growth model by Francois and Shi (1999) where labor rather than capital is the sole input for innovations and for
intermediate goods production, cycles can arise endogenously from contemporaneous complementarities between investors devoting labor to innovation for temporary profits.

The cyclical fluctuations in consumption, investment, innovation and growth represent another source of efficiency loss, given diminishing marginal utility of consumption and diminishing marginal product of factors. This adds to the efficiency losses of monopoly pricing analyzed in the literature, which differs from the Pareto optimal variations in consumption across time and sectors facing exogenous shocks found in Long and Plosser (1983). Therefore, important macroeconomic questions arise as follows. Can government policies mitigate or eliminate such cyclical fluctuations and promote innovation and growth at the same time? Can such government policies, if they exist, enhance social welfare?

When attempting to answer these questions, it is natural to focus on subsidization associated with R&D activities or with the purchase of new R&D products. Although such subsidization has been considered in the literature with innovation and growth as mentioned earlier, the existing studies have mainly focused on how the subsidies can mitigate the static and dynamic losses in efficiency of monopoly pricing on a stable balanced growth path. To the best of our knowledge, the only work on subsidization in the unified model so far is Aloi and Lasselle (2007), in which a lump-sum subsidy to innovators financed by a lump-sum tax on consumers can promote growth, stabilize the innovation cycles and increase welfare. However, they essentially allow the use of current tax revenue to form available capital for current innovation via subsidization: With a balanced budget between tax revenue and subsidy spending, their subsidy takes the form of giving capital to innovators and adds to available capital in the resource constraint for innovation and intermediate goods production.
Their assumption is at odds with the spirit of time-to-build in neoclassical capital accumulation. In the Matsuyama model, the constraint on available capital from previous saving for current innovation and intermediate goods production is a necessary assumption for the creation of cycles. It remains to show whether and how more realistic subsidization by taxation, that does not relax this assumption, can induce changes in individuals' behavior to stabilize the balanced growth path for welfare gains.

In this paper, we explore whether flat-rate subsidization associated with the profitability of R&D activities can mitigate or eliminate cycles for welfare gains in the Matsuyama model. In doing so, the subsidy provides additional awards to innovators without relaxing the constraint on available capital for innovation and intermediate goods production. We find that subsidies to R&D investment or to the purchase of newly invented intermediate goods, financed by a consumption tax, can arbitrarily reduce the threshold level of capital per variety, beyond which the economy moves from the investment phase to the innovation phase. Moreover, sufficient subsidization can change the balanced growth path from an unstable one to a stable one and thus eventually eliminate the cycles. In numerical examples, such subsidies can achieve substantial welfare gains equivalent to as much as a 15% increase in consumption in every period; the optimal subsidy rates are found indeed in the range that lead to either oscillatory or monotonic convergence toward the stable steady state of capital per variety, or the balanced growth path.

Our results are consistent with the postwar experiences in some industrial nations such as the United States where substantial subsidies are provided to R&D activities and to the purchase of new equipment. For example, there is substantial subsidization in the US tax
system: a 50% immediate writing-off of equipment investment, expensing of R&D expenditures, and accelerated depreciation allowances, according to Gordon, Kalambokidis, Rohaly and Slemrod (2004), Gordon, Kalambokidis and Slemrod (2004) and others. At the same time these countries observe much more innovations but dampened recessions compared to previous times on average.

The rest of the paper proceeds as follows. Section 2 introduces the building blocks of the model. Section 3 characterizes the steady states in different regimes and analyzes the global dynamics for different levels of subsidization. Section 4 deals with optimal subsidy rates and presents numerical simulation results. Section 5 concludes.

2. The model

The model is an extension from Matsuyama (1999) by considering subsidies to R&D spending and to the purchase of newly invented intermediate goods, financed by a consumption tax. Time is discrete from one to infinity.

2.1. The structure of production and innovation

There is a single final good taken as a numeraire; it is produced competitively using capital and labor; it can either be consumed or invested. Labor is supplied inelastically at an amount $L$ that also stands for the size of the working population. Let $K_{t-1}$ denote the capital stock saved in period $t-1$ starting with an initial capital stock $K_0 > 0$; it is used for the production of existing intermediates or for the innovation of new intermediates in period $t$. This means that capital takes one period to become productive in the spirit of time-to-build
capital. Without this realistic assumption, the economy would always be on the unique balanced growth path as in the earlier R&D growth models.

Capital is first converted into a composite of intermediate goods by a symmetric CES function. Let $x_t(z)$ denotes the $z$th type of available intermediate good in the range $[0, N_t]$ in period $t$. Labor and the composite of intermediates are combined through a Cobb-Douglas technology for final goods production:

$$Y_t = A(L)^{1/\sigma} \left\{ \int_0^{N_t} [x_t(z)]^{1-1/\sigma} \, dz \right\},$$

(1)

where $A > 0$ is the total factor productivity parameter, and $\sigma > 1$ is the direct partial elasticity of substitution between every pair of intermediate goods.

One unit of each type of intermediate good is manufactured by converting $a$ units of capital. In each period $t$, old intermediate goods in the range $z \in [0, N_{t-1}]$ starting with $N_0 > 0$ are sold competitively, while new intermediate goods of variety $z \in [N_{t-1}, N_t]$ may be introduced via innovation and sold exclusively by their innovators in period $t$. Innovating a new intermediate requires $F$ fixed units of capital. Because all intermediate goods enter final goods production symmetrically, we have $x_t(z) \equiv x_t^c$ for $z \in [0, N_{t-1}]$, and $x_t(z) \equiv x_t^m$ for $z \in [N_{t-1}, N_t]$. Then the profit function for firms in the final goods sector can be expressed as:

$$\Pi_t = A(L)^{1/\sigma} \left[ N_{t-1} (x_t^c)^{1-1/\sigma} + (N_t - N_{t-1}) (x_t^m)^{1-1/\sigma} \right]$$

$$-N_{t-1} p_t^c x_t^c \left( 1 - s_x \right) (N_t - N_{t-1}) p_t^m x_t^m - w_t L, \quad 0 \leq s_x < 1,$$

(2)

where $p_t^c$ and $p_t^m$ are the prices of old and new intermediate goods, respectively; $s_x$ is the constant subsidy rate to the purchase of new intermediate goods used in Barro and Sala-i-Martin (1995) and Zeng and Zhang (2007) but not considered in Aloi and Lasselle.
(2007); and \( w_f \) is the wage rate per unit of labor. According to Zeng and Zhang (2007), the subsidies to final output or to the purchase of intermediate goods are equivalent concerning their effects on growth and welfare. Thus, we only consider the latter.

In the final goods sector, factors are paid by their marginal products:

\[
p_c^f = (1 - 1/\sigma) A(L)^{1/\sigma} \left( x_c^f \right)^{-1/\sigma},
\]

\[
p_c^m (1 - s_x) = (1 - 1/\sigma) A(L)^{1/\sigma} \left( x_c^m \right)^{-1/\sigma},
\]

\[
w_f = \frac{1}{\sigma} \frac{Y_f}{L}.
\]

Let \( r \) denote the price of capital. Then the marginal cost of manufacturing intermediate goods in period \( t \) is equal to \( ar \). All old intermediate goods are supplied competitively at a price level equal to the marginal cost, \( p_t(z) = p_c^e = ar \) for \( z \in [0, N_{t-1}] \), while all new intermediate goods, once introduced, are sold monopolistically at a higher price level, \( p_t(z) = p_c^m = [\sigma/(\sigma - 1)]ar > p_c^e \), for \( z \in [N_{t-1}, N_t] \), following from (4). From equations (3) and (4), the relationship between \( x_c^e \) and \( x_c^m \) must satisfy:

\[
\frac{x_c^e}{x_c^m} = \left( \frac{p_c^e}{p_c^m (1 - s_x)} \right)^{\sigma} = \left[ 1 - \frac{1}{\sigma} \right]^{\sigma} (1 - s_x)^{\sigma}.
\]

Absent subsidies, the higher price of new intermediates than that of old intermediates yields a smaller equilibrium quantity of each new intermediate than that of each old intermediate, \( x_c^m < x_c^e \). This asymmetry in the equilibrium quantities of new versus old intermediates must lead to static and dynamic efficiency losses. The lower demand for new than for old intermediate goods leads to a dynamic efficiency loss in terms of decelerating the rate of innovation because it reduces the profitability scale for innovators to recover the fixed R&D cost. At the same time, the lower demand for new than for old intermediate...
goods leads to a static efficiency loss in terms of decreasing final output because all intermediates enter final goods production symmetrically and have diminishing marginal contributions to final output.

Subsidies on the purchase of new intermediate goods strengthen final goods producers' demand for new intermediate inputs relative to old ones by reducing the user cost of new intermediates. According to (6), when \( s_s < 1/\sigma \), \( x_c > x_m \); when \( s_s = 1/\sigma \), \( x_c = x_m \); when \( s_s > 1/\sigma \), \( x_c < x_m \). Thus, the subsidy on the purchase of new intermediates may affect the dynamic system significantly in this model.

The one period monopoly enjoyed by the innovator makes it possible to recover the fixed R&D cost. There is free entry for innovative activities. The monopoly profit is equal to the sales revenue net of the fixed R&D cost and the variable manufacturing cost:

\[
\pi_t = p_t x_t^m - r_t \left[ ax_t^m + \left( 1 - s_n \right) F \right].
\]

Here, \( s_n \) is the constant subsidy rate to the fixed R&D cost \( r_t F \) used in Barro and Sala-i-Martin (1995) and Zeng and Zhang (2007), and differs from the lump-sum subsidy used in Aloi and Lasselle (2007). The free entry ensures the following in equilibrium

\[
ax_t^m \leq (\sigma - 1) \left( 1 - s_n \right) F, \quad N_t \geq N_{t-1}, \quad \left[ ax_t^m - (\sigma - 1) \left( 1 - s_n \right) F \right] (N_t - N_{t-1}) = 0.
\]

That is, when potential innovators expect the sale of a new intermediate good to be smaller than the break-even point (i.e. \( x_t^m < (\sigma - 1) \left( 1 - s_n \right) F / a \)), there is no incentive for innovation, thereby \( N_t = N_{t-1} \). In equilibrium with free entry, when innovation occurs (i.e. \( N_t > N_{t-1} \)), the innovator must just break even such that \( ax_t^m = (\sigma - 1) \left( 1 - s_n \right) F \). By lowering the cost of innovation virtually to any level, the subsidy on the R&D cost can reduce this break-even level of the demand for a new intermediate virtually to any level as well, and may therefore
have significant effects on the dynamic system of the model.

Regardless of the value of the subsidy rates, the resource constraint on the use of available capital for intermediate goods production and innovation in period $t$ is:

\[ K_{t-1} = N_{t-1} ax_t + (N_t - N_{t-1})(ax_m + F). \]  

(8)

This differs from the assumption in Aloi and Lasselle (2007) that regards the subsidy as an addition to available capital.

Substituting equations (6) and (7) into the above constraint leads to

\[ ax_t = \alpha \left[ 1 - \frac{1}{\sigma} \right] (1 - s_y) \theta \sigma F \min \left\{ k_{t-1}, (1 - s_y) \theta \sigma (1 - s_n) \right\}, \]  

(9)

\[ N_t = N_{t-1} + \max \left\{ 0, \frac{K_{t-1}}{\left[ \sigma - s_n (\sigma - 1) \right] F} - (1 - s_y) \theta \sigma (1 - s_n) \right\}, \]  

(10)

where

\[ k_i \equiv \frac{K_i}{\left( \theta \sigma F \right) N_i}, \quad \theta = \left[ 1 - \frac{1}{\sigma} \right]^{1-\sigma}, \quad \sigma \in [1, e], \quad e = 2.71828.... \]

Clearly, for innovators to break even in period $t$, the initial capital stock $K_{t-1}$ must be abundant enough relative to variety $N_{t-1}$. According to (9), increasing the subsidy rate on the fixed R&D cost will reduce the demand for both new and old intermediates, while increasing the subsidy rate on purchasing new intermediates will reduce the demand for old intermediates, given any initial state $(N_{t-1}, K_{t-1})$ such that $k_{t-1}$ is large enough for innovators to break even. According to (10), increasing either of the two subsidy rates will increase the rate of innovation, given any initial state such that $k_{t-1}$ is large enough for innovators to break even.

We can now rewrite equation (1) as
Given any initial state that allows innovators to break even, the subsidies can increase final output by promoting innovation for faster variety expansion, but can reduce final output by reducing the demand for each intermediate good. However, the subsidy on the purchase of new intermediates can increase final output by increasing the demand for each new intermediate unless it is too large. To detail such effects further, we rewrite equation (11) as the following, using equations (7), (8), (9) and (10):

\[ Y_t = A \left( LN_{t-1} \right) ^{1/\sigma} \left( \frac{K_{t-1}}{a} \right) ^{1-1/\sigma}, \quad \text{if} \quad k_{t-1} \leq k_c \equiv (1-s_x)^\sigma (1-s_n); \]

\[ Y_t = A \left( L \right) ^{1/\sigma} \left\{ \frac{K_{t-1}}{\sigma - s_n (\sigma - 1)} \right\} \left[ \frac{(\sigma - 1) F (1 - s_n)}{a} \right] ^{1-1/\sigma} \]

\[ + N_{t-1} (1-s_x)^{1-1/\sigma} \left[ \frac{s_n + s_x (1 - s_n)}{\sigma - s_n (\sigma - 1)} \right] \left[ \frac{\theta \sigma F (1 - s_n)}{a} \right] ^{1-1/\sigma} \}, \quad \text{if} \quad k_{t-1} \geq k_c. \]  

The critical value of the capital-variety ratio, \( k_c = (1-s_x)^\sigma (1-s_n) \), below which there is no innovation and hence no subsidization by construction, divides government action in this model into \textit{policy-dormant} and \textit{policy-active} regions, respectively. Interestingly and intuitively, this threshold level of the capital-variety ratio is decreasing with each of the subsidy rates \( s_x \) and \( s_n \). In particular, increasing the rate of either subsidy can reduce the threshold level of the capital-variety ratio, \( k_c \), virtually to anywhere above zero, enhancing the chance for the economy to stay in the policy-active region with R&D activities. Thus, the subsidization may significantly change the dynamic path of the model.

According to (12), given an initial state \( (N_{t-1}, K_{t-1}) \) such that \( k_{t-1} \geq k_c \), subsidizing either the fixed R&D cost or the purchase of new intermediates can increase final output if the subsidy rates are sufficiently low, when their positive effect on variety expansion...
dominates. The opposite occurs for further increases in the subsidy rates if the subsidy rates are already sufficiently high, when their negative effect on the demand for intermediates dominates. To see this clearly, we differentiate final output with respect to one subsidy rate at a time for any initial state \((N_{t-1}, K_{t-1})\) such that \(k_{t-1} \geq k_c\). Focusing first on how \(s_n\) affects \(Y_t\) at \(s_x = 0\) and \(k_{t-1} \geq k_c\), \(dY_t/ds_n\) is signed by two parts additively. One part with the derivative of \((1-s_n)^{1-1/\alpha}/[\sigma-s_n(\sigma-1)]\) with respect to \(s_n\) is signed by \(-s_n\), while the other part with the derivative of \(s_n(1-s_n)^{1-1/\alpha}/[\sigma-s_n(\sigma-1)]\) is signed by \(1-s_n(2-1/\sigma)\). Combining the two parts together, \(dY_t/ds_n\) must be positive for very small \(s_n\) but becomes negative when \(s_n\) becomes larger, at least when \(s_n > 1/(2-1/\sigma)\).

When focusing on \(s_x\) at \(s_n = 0\), \(dY_t/ds_x\) is signed by \(1-\sigma s_x\) through signing the derivative of \((1-s_x)^{\sigma^{-1}}s_x\) with respect to \(s_x\). Thus, final output increases with \(s_x\) when \(s_x < 1/\sigma\) under which \(x_c > x_m\); final output peaks at \(s_x = 1/\sigma\) whereby \(x_c = x_m\): any further increase in \(s_x\) leads to \(x_c < x_m\) and thus reduces final output. Therefore, given the initial state, the effects of the subsidies on final output also alter the static efficiency of the model.

Equations (10) and (12) are simplified to

\[
\frac{N_t}{N_{t-1}} = \psi(k_{t-1}, s_x, s_n) = \max \left\{ 1, \frac{\theta \sigma}{\sigma - s_n(\sigma - 1)} \left[ k_{t-1} - (1-s_n)^{\sigma} (1-s_n) \right] \right\}, \quad (13)
\]

\[
\frac{Y_t}{K_{t-1}} = \phi(k_{t-1}, s_x, s_n) = A \left( \frac{L}{\theta \sigma F} \right)^{1/\alpha} (k_{t-1})^{-1/\alpha}, \text{ if } k_{t-1} \leq k_c; \text{ otherwise,}
\]

\[
\phi(k_{t-1}, s_x, s_n) = A \left( \frac{L}{\theta \sigma F} \right)^{1/\alpha} \left(1-s_n\right)^{1-1/\alpha} \left[ 1 + \frac{(1-s_n)^{\sigma-1}}{k_{t-1}} \frac{s_n}{\sigma + s_x(1-s_n)} \right]. \quad (14)
\]

To economize, we assume \(L = 1, a = 1, \text{ and } F = 1/\theta \sigma\) without changing the essence of
the results.

2.2. The households’ problem and the government’s balanced budget

The economy is populated by overlapping generations of equal size $L$ (normalized to one). Agents are identical and live for two periods, working when young and living in retirement when old. At every period $t$, a new generation of workers enters the economy. Each worker provides one unit of labor inelastically and earns labor income $w_t$. Part of the income is consumed when young, $C^1_t$, and the rest is saved for their old-age consumption, $C^2_{t+1}$. We assume that the government uses a flat-rate consumption tax, $\tau_{c,t}$, to finance its subsidy expenditure, and runs a balanced budget in every period.

For tractability, the preference of young agents in generation $t$ is assumed as

$$U' = (1-s) \ln C^1_t + s \ln C^2_{t+1}, \quad 0 < s < 1,$$

where $(1-s)$ and $s$ are taste parameters attached to young-age and old-age consumption, respectively. The lifetime budget constraint of the representative worker is

$$\left(1+\tau_{c,t}\right)C^1_t + \left(1+\tau_{c,t+1}\right)C^2_{t+1}/r_{t+1} = w_t,$$

where $r_t$ is the period $t$ interest factor.

In every period, young agents save optimally to maximize utility in (15) subject to (16), yielding a simple saving function:

$$S_t = s w_t = (s/\sigma) Y_t.$$

The model can also be extended into an infinitely lived agent framework as shown in Matsuyama (2001) where period-2 cycles exist with a logarithmic utility function. The overlapping-generations framework used here renders simplicity and makes our results more
comparable to those in Matsuyama (1999).

The asset market clearing condition is \( K_i = Y_i - C_i^1 - C_i^2 \) – government spending. With a balanced budget in every period, government spending on subsidization is equal to the tax revenue \( \tau_{ct}(C_i^1 + C_i^2) \). We can therefore rewrite the asset market clearing condition as:

\[ K_i = Y_i - (1+\tau_{ct})C_i^1 -(1+\tau_{ct})C_i^2. \]

All income goes to households according to \( w_i = (1/\sigma)Y_i \) and \( S_{t+1}r_t = (1-1/\sigma)Y_i \); it is spent on life-cycle consumption in such an optimal way

\[
C_i^1 = \frac{w_i - sw_i}{1+\tau_{ct}} = \frac{(1-s)w_i}{1+\tau_{ct}} = \left(\frac{1-s}{1+\tau_{ct}}\right)Y_i
\]

\[
C_i^2 = \frac{S_{t+1}r_t}{1+\tau_{ct}} = \frac{K_{t+1}r_t}{1+\tau_{ct}} = \left(\frac{1-1/\sigma}{1+\tau_{ct}}\right)Y_i.
\]

It is important to observe that a higher tax rate in this model creates a switch from private consumption spending to government subsidization spending as awards to innovation.

Substituting the consumption function above into the asset market clearing condition yields

\[ K_i = Y_i - (1-s)\frac{1}{\sigma}Y_i - (1-1/s)Y_i \]

\[ = \frac{S}{\sigma}Y_i.\]

That is, investment in capital is equal to private savings in every period in this model:

\[ K_i = S_i. \]

From equations (13), (14), (17) and (18), the dynamics of the economy can be uniquely determined by the following system of first-order difference equations in \( K \) and \( N \):

\[ K_i = \left(\frac{s}{\sigma}\right)\phi(k_{t-1},s,s_n)K_{t-1}, \]

(19a)

\[ N_i = N_{t-1} + \max \left\{ 0, \frac{\theta}{1-s_n(1-1/\sigma)}\left[K_{t-1} - (1-s_s)^\sigma (1-s_n)N_{t-1} \right] \right\}, \]

(19b)

for an initial state at time 1, \((K_0,N_0)\), and exogenously given constant subsidy rates, \(s_s\).
and $s_n$. Observe that sufficient subsidization of either type can lead to the introduction of new intermediates, $N_i - N_{i-1} > 0$, for any initial state at time $t$, $(N_{i-1}, K_{i-1})$.

With the solution for consumption, innovation and intermediate goods, the government's balanced budget in every period links the equilibrium tax rate to subsidy rates:

$$\tau_{c,t} \left( C^*_t + C^*_t \right) = s_k F r_t (N_t - N_{t-1}) + s_x a_r t (\sigma - 1) (N_t - N_{t-1}) x^m.$$  \quad (20)

In (20), the left-hand side is the total revenue from taxing co-existing young and old generations’ consumption, while the right-hand side is the total expenditure on subsidizing the R&D cost and the purchase of new intermediate products. When no innovation occurs (i.e., $N_t = N_{t-1}$), the consumption tax equals zero and accordingly the economy is in the policy-dormant regime. Aloi and Lasselle (2007) also assume a balanced budget between current subsidy spending and current tax revenue. But because their subsidy adds to available capital directly, they essentially allow the use of current tax revenue to form available capital for current innovation and intermediate goods production. By contrast, we do not allow current tax revenue to relax the constraint on available capital for innovation and intermediate goods production via subsidization. The justification for our assumption is that, as mentioned earlier, this constraint on available capital for innovation and intermediate goods production is the necessary assumption for endogenous cycles in the Matsuyama model to differentiate it from the AK-style R&D growth models without cycles.

Combining the consumption functions given earlier together with (7) and (10), the equilibrium tax rate in (20) can be determined by the following

$$\frac{\tau_{c,t}}{1 + \tau_{c,t}} \left( \frac{1 - s_n}{\sigma} \right) = \max \left\{ 0, \left( 1 - \frac{1}{\sigma} \right) s_k + \sigma s_n (1 - s_n) \left( 1 - \frac{k_t}{k_{t-1}} \right) \right\}.$$  

Using this together with (19b), the equilibrium tax rate can also be expressed as a reduced
form solution given an initial state and the subsidy rates:

$$
\tau_{c,t} = \max \left\{ 0, \frac{k_{t-1} - k_c}{\sigma - s_n \left( \frac{\sigma - s_n (\sigma - 1)}{\sigma - 1 + \sigma s_n (1 - s_n)} \right), (k_{t-1} - (k_{t-1} - k_c))} \right\}
$$

(21)

According to this, the equilibrium consumption tax rate is positive only when the economy falls into the policy-active region, i.e. $k_{t-1} > k_c$.

3. The steady state and global dynamic analysis

From equations (19a) and (19b), the law of motion for the capital variety ratio, $k_t$, is governed by the following one-dimensional mapping, $\Phi : R_+ \rightarrow R_+$,

$$
k_t = \Phi(k_{t-1}) = \begin{cases} 
(sA/\sigma) \left( k_{t-1} \right)^{1-\sigma} & \text{if } k_{t-1} \leq k_c \\
1 - s_n (1 - 1/\sigma) + \Theta \left[ k_{t-1} - (1 - s_n) \right]^{\sigma} (1 - s_n) & \text{if } k_{t-1} \geq k_c 
\end{cases}
$$

(22)

Recall that the critical value of $k$, below which there is no innovation, is given by $k_c = (1 - s_n)^\sigma (1 - s_n)$.

The steady state of the economy is defined as an equilibrium path on which $k_t = K_t/N_t$ stays constant over time for any given constant subsidy rates, $s_s$ and $s_n$. According to (22), the steady state of the dynamic system is uniquely determined. Whether new intermediates are introduced or not at the steady state of capital per variety depends on the relationship between the steady state of $k$ and the critical value $k_c$.

First, if $k_t = k^* \leq k_c$ in a steady state, then according to (13) and (14), $N_t = N_{t-1}$ and $K_t = K_{t-1}$. In this steady state, there is no innovation; all the intermediate goods are competitively supplied; and the economy does not grow in the long run. From (19a), on this...
neoclassical stationary path, \( k^* \equiv (sA/\sigma)^\sigma \). The existence of such a stationary path requires that \( sA/\sigma \leq (k_c)^{1/\sigma} \).

Now, suppose that \( k_i = k^{**} > k_c \) holds in a steady state. From (19b), the balanced growth path satisfies the following:

\[
\frac{s}{\sigma} \phi(k_{i-1}, s_i, s_n) = \frac{K_i}{N_i} = 1 + \frac{\theta \sigma}{\sigma - s_n (\sigma - 1)} \left[ k_{i-1} - (1 - s_i) \sigma (1 - s_n) \right] > 1.
\]

In this steady state, the capital stock of the economy is large enough relative to the number of existing intermediates such that new intermediates are introduced and that \( K_i \) and \( N_i \) share the same growth rate. The existence of such a balanced growth path requires \( (s/\sigma) \phi(k_{i-1}, s_i, s_n) > 1 \).

These results concerning the steady state of the dynamic system in (22) are given below.

**Proposition 1.** Let \( G \equiv (sA/\sigma) \left[ (1 - s_i) (1 - s_n) \right]^{1/\sigma} \), with \( 0 \leq s_i < 1 \) and \( 0 \leq s_n < 1 \).

1. If \( G \leq 1 \), the dynamic system has a unique steady state \( k^* = \Phi(k^*) \) where

\[
k^* = (sA/\sigma)^\sigma \leq k_c = (1 - s_i)^\sigma (1 - s_n).
\]

At this steady state, the economy has no innovation and does not grow.

2. If \( G > 1 \), the dynamic system has a unique steady state \( k^{**} = \Phi(k^{**}) \) where

\[
k^{**} = \left[ \theta (1 - s_i)^\sigma (1 - s_n) + s_n (1 - 1/\sigma) - 1 + (sA/\sigma) (1 - s_n)^{1/\sigma} + \Delta^{1/2} \right]/(2 \theta) > k_c,
\]

\[
\Delta = \left[ 1 - s_n(1 - 1/\sigma) - \theta (1 - s_i)^\sigma (1 - s_n) - (sA/\sigma) (1 - s_n)^{1/\sigma} \right] \sigma
\]

\[
+ \theta (sA/\sigma) (1 - s_n)^{1-\sigma} (1 - s_i)^{\sigma-1} \left[ s_n/\sigma + s_i (1 - s_n) \right].
\]

At this steady state, the economy grows in \( (N_i, K_i) \) at the same constant rate

\[
g = \frac{s}{\sigma} \phi(k^{**}, s_i, s_n) = 1 + \frac{\theta \sigma}{\sigma - s_n (\sigma - 1)} \left[ k^{**} - (1 - s_i)^\sigma (1 - s_n) \right].
\]

17
Proof. The solutions for the steady state $k^*$ or $k^{**}$ follow the respective scenarios in (22).

What remains to show is $k^{**} > k_c$ for $G > 1$. First, note the following implication of $G > 1$:

$$
\Delta = [1-s_n(1-1/\sigma)-(sA/\sigma)(1-s_n)^{1-\alpha}]^2 + 4\theta (sA/\sigma) k_c (1-s_n)^{-\alpha} (1-s_x)^{-1} \\
[\sigma (1-s_n)^2 + s_n/\sigma ]
$$

$$
= \theta^2 k_c^2 + [1-s_n(1-1/\sigma)-(sA/\sigma)(1-s_n)^{1-\alpha}]^2 - 2\theta k_c [1-s_n(1-1/\sigma)-(sA/\sigma)(1-s_n)^{1-\alpha} (1-s_x)^{-1} [s_n(1-s_n)+s_n/\sigma ]
$$

Note that $1-s_n(1-1/\sigma)-(1-s_x)(1-s_n)=s_n(1-s_n)+s_n/\sigma \geq 0$ and that $G > 1$ implies $(sA/\sigma)(1-s_n)^{-1/\alpha} > (1-s_x)(1-s_n)$. We can now rewrite the expression of $\Delta$ below:

$$
\Delta = \theta^2 k_c^2 + [1-s_n(1-1/\sigma)-(sA/\sigma)(1-s_n)^{1-\alpha}]^2 + 2\theta k_c [2(sA/\sigma)(1-s_n)^{-1/\alpha} (1-s_x)^{-1} \\
[\sigma (1-s_n)^2 + s_n/\sigma ] - 1 + s_n(1-1/\sigma) + (sA/\sigma)(1-s_n)^{-1/\alpha}
$$

$$
> \theta^2 k_c^2 + [1-s_n(1-1/\sigma)-(sA/\sigma)(1-s_n)^{1-\alpha}]^2 + \\
2\theta k_c [1-s_n(1-1/\sigma)-(1-s_x)(1-s_n)]\left[2(sA/\sigma)(1-s_n)^{-1/\alpha} (1-s_x)^{-1} - 1\right]
$$

$$
> \theta^2 k_c^2 + [1-s_n(1-1/\sigma)-(sA/\sigma)(1-s_n)^{1-\alpha}]^2 + \\
2\theta k_c [1-s_n(1-1/\sigma)-(1-s_x)(1-s_n)]
$$

$$
> \theta^2 k_c^2 + [1-s_n(1-1/\sigma)-(sA/\sigma)(1-s_n)^{1-\alpha}]^2 + \\
2\theta k_c [1-s_n(1-1/\sigma)-(1-s_x)(1-s_n)]
$$

$$
= \theta k_c + 1-s_n(1-1/\sigma)-(sA/\sigma)(1-s_n)^{1-\alpha} + \Delta^{1/2} \right] (2\theta) > 2 \theta k_c \right) (2\theta) = k_c. 
$$

So $k^{**} = \theta k_c + 1-s_n(1-1/\sigma)-(sA/\sigma)(1-s_n)^{1-\alpha} + \Delta^{1/2} \right]$ (2\theta) > 2 \theta k_c \right) (2\theta) = k_c$. The other root, $k^{**} = \theta (1-s_x)^{\alpha} (1-s_n)+s_n(1-1/\sigma)-1+(sA/\sigma)(1-s_n)^{-1/\alpha} - \Delta^{1/2} \right] (2\theta) \right), is dropped for being inconsistent with $k^{**} > k_c$. Q.E.D.

The implication of Proposition 1 is as follows. First, whether the economy grows or not in the steady state depends on both the fundamentals, such as the saving rate, the productivity coefficient, the degree of substation between intermediate goods, and the subsidy rates. Other fundamentals being equal, the higher the rates of both subsidies, the more likely the economy moves beyond the critical $k_c$ toward the balanced growth path. In
fact, given \( k^* = (sA/\sigma)^\sigma \), sufficient subsidization can ensure that \( k^* > k_c = (1-s_x)^\sigma (1-s_n) \), while \( k^* > k_c = (1-s_x)^\sigma (1-s_n) \) remains valid for all permissible rates of subsidies in the full range of \([0,1)\). That is, sufficient subsidization can rule out the neoclassical steady state in the long run and replace it by the steady state with balanced growth in capital and the variety of intermediates. Furthermore, the steady state value of \( k \) is uniquely determined in each regime with or without innovation, depending on the fundamentals.

Now, we investigate the stability of the steady state by examining the asymptotic behavior of \( k = K_t/N_t \), from any arbitrary initial state \( k_0 = K_0/N_0 > 0 \). The mapping \( k_t = \Phi(k_{t-1}) \) in equation (22) is continuous: It is increasing in the range of \((0,k_c)\) and it may be increasing or decreasing in the range of \((k_c,\infty)\).

When \( k_{t-1} \leq k_c = (1-s_x)^\sigma (1-s_n) \), there is no innovation in period \( t \), that is, \( N_t = N_{t-1} \). In this region without innovation, all the intermediates are sold competitively and economic growth is led solely by capital accumulation with diminishing returns. Consequently, the two kinds of innovation-oriented subsidies are non-operative in this neoclassical growth regime without innovation.

On the other hand, when \( k_{t-1} > k_c = (1-s_x)^\sigma (1-s_n) \), new intermediates are introduced and both subsidies can be operative. Economic growth in this region is led by both the accumulation of capital and the innovation of new varieties of intermediates. Intuitively, when the growth rate of capital accumulation dominates the growth rate of the variety of intermediates, the resultant ratio of capital per variety \( k_t = K_t/N_t \) will increase; conversely, it will decrease. The slope of the transition curve of \( k_t = \Phi(k_{t-1}) \) in this region with innovation plays a crucial role in determining the asymptotic behavior of \( k_t \) and thus
deserves careful investigation.

Without the use of subsidies at \( s_n = s_n = 0 \), the dynamics of \( k_r = \Phi(k_{r-1}) \) in (22) will become exactly the same as in Matsuyama (1999), where \( k_c = 1 \), and the mapping for \( k_r \) is always decreasing in the policy-active region with innovation. Under the empirically plausible conditions \( 1 < G < \theta - 1 \) in his benchmark model, period-2 cycles are prevalent when \( k_r \) alternates between the two regions forever. With the subsidies, it is important to ask whether the subsidies can change the slope of the transition equation \( k_r = \Phi(k_{r-1}) \) so as to mitigate or even eliminate the cycles. The answer is given below under the condition \( \theta > 2 \), a scenario whereby period-two cycles emerge and persist forever without subsidization.

**Proposition 2.** Suppose \( \theta > 2 \). Define \( G_0 = sA/\sigma \) and \( G = \left( sA/\sigma \right) \left[ (1-s_n)(1-s_n)^{1/\sigma} \right] \)

with \( 0 \leq s_n, s_s < 1 \).

1. If \( G \leq 1 \), then, for any given \( k_0 \in R_+ \), the economy will eventually converge toward a neoclassical stationary path with \( \lim_{t \to \infty} k_r = k^* \) and settle down in the policy-dormant region.

2. If \( G > 1 \) and \( s_s \leq 1/\sigma \), for any given \( k_0 \in R_+ \), there may exist cycles forever if the subsidy rates are low enough; if the subsidy rates are high enough (e.g. \( s_n > \sigma (\theta - 2)/[1 + \sigma (\theta - 2)] \) at \( s_s = 0 \) or \( s_s \to 1/\sigma \) at \( s_n = 0 \)), then \( |dk_r/dk_{r-1}| < 1 \) and the economy will converge toward the balanced growth path oscillatorily with \( \lim_{t \to \infty} k_r = k^{***} \).

3. If \( G > 1 \) and \( s_s > 1/\sigma \), for any given \( k_0 \in R_+ \), the economy will converge toward the
balanced growth path monotonically with \[ \lim_{t \to \infty} k_t = k^* \].

**Proof.** In case (1) with \( k^* = (sA/\sigma)^\sigma < k_\epsilon = (1-s_\epsilon)^\sigma (1-s_n) \) and \( k_{t-1} < k_\epsilon \), the slope of the transition equation \( k_t = \Phi(k_{t-1}) = (sA/\sigma)k_{t-1}^{1-\alpha} \) in (22) is always positive, exceeding 1 at the origin \( (k_{t-1} \to 0) \) and falling below 1 at the steady state \( (k_{t-1} = k^* = (sA/\sigma)^\sigma) \) according to:

\[
\frac{dk_t}{dk_{t-1}} = (1-1/\sigma)(sA/\sigma)k_{t-1}^{1-\alpha}
\]

because \( \sigma > 1 \). The steady state level \( k^* < k_\epsilon \) is thus stable and the sequence \( \{k_t\}_{t=0}^\infty \) converges toward \( k^* \) for any \( k_0 \) as in the standard neoclassical growth model. We illustrate case (1) in Figure 1.

In cases (2) and (3) with \( k^* = (sA/\sigma)^\sigma > k_\epsilon = (1-s_\epsilon)^\sigma (1-s_n) \), the slope of the transition equation \( k_t = \Phi(k_{t-1}) \) in (22) for \( k_{t-1} > k_\epsilon \) is derived as

\[
\frac{dk_t}{dk_{t-1}} = \frac{(sA/\sigma)(1-s_n)^{1-\alpha} \left[ 1-s_n \left( 1-1/\sigma \right) \right] [1-\Theta(1-s_n)^{\sigma-1}]}{\left[ 1-s_n \left( 1-1/\sigma \right) + \Theta \left( k_{t-1} - (1-s_n)^\sigma (1-s_n) \right) \right]^2}.
\]

Here, \( 1-s_n(1-1/\sigma) > 0 \) because \( s_n \in [0,1) \) and \( \sigma > 1 \). Also, \( k_{t-1} - (1-s_n)^\sigma (1-s_n) > 0 \) for \( k_{t-1} > k_\epsilon \). So the sign of \( dk_t/dk_{t-1} \) is the same as the sign of \( 1-\Theta(1-s_n)^{\sigma-1} \). Recalling \( \Theta = (1-1/\sigma)^{1-\sigma} > 1 \) under \( \sigma > 1 \), we have: sign \( dk_t/dk_{t-1} > 0 \) if and only if \( 1 > s_\epsilon > 1/\sigma \) because with \( s_\epsilon \in [0,1) \), \( 1-\Theta(1-s_\epsilon)^{\sigma-1} > 0 \) corresponds to \( 1 > s_\epsilon > 1/\sigma \). Accordingly, \( dk_t/dk_{t-1} \leq 0 \) if and only if \( 0 \leq s_\epsilon \leq 1/\sigma \) under which \( 1-\Theta(1-s_\epsilon)^{\sigma-1} \leq 0 \). Also, the absolute value of \( dk_t/dk_{t-1} \) is monotonically decreasing in \( k_{t-1} \), and approaches zero when \( k_{t-1} \) approaches infinity, implying that the dynamic system in (22) cannot converge outward to infinity.

For \( k_{t-1} > k_\epsilon \), there are thus two possibilities with either \( dk_t/dk_{t-1} \leq 0 \) or
$dk_t/dk_{t-1} > 0$. If $0 \leq s_x \leq 1/\sigma$ and thus $dk_t/dk_{t-1} \leq 0$ beyond $k_e$, then the economy may either oscillate forever with cycles or eventually converge toward the steady state of the balanced growth path such that $\lim_{t \to \infty} k_t = k^{**}$, depending on whether $|dk_t/dk_{t-1}| \geq 1$ or $< 1$. Specifically, if the subsidy rates are low enough (say zero), then $|dk_t/dk_{t-1}| > 1$ prevails under $\theta > 2$ and the economy behaves as in the original model of Matsuyama (1999) with endogenous cycles forever. If the subsidy rates are high enough, then we show $|dk_t/dk_{t-1}| < 1$ for $dk_t/dk_{t-1} \leq 0$ as follows. First, the slope $dk_t/dk_{t-1}$ of $k_t = \Phi(k_{t-1})$ at $k_{t-1} = k^{**}$ can be rewritten as:

$$
\frac{dk_t}{dk_{t-1}} \bigg|_{k_{t-1} = k^{**}} = \frac{(sA/\sigma)(1-s_n)^{1/\alpha}[1-s_x(1-1/\sigma)](1-\theta(1-s_x)^{\sigma-1})}{1-s_n(1-1/\sigma)+\theta[k^{**}-(1-s_x)^{\sigma}(1-s_n)]^2}
$$

$$
= \frac{4(sA/\sigma)(1-s_n)^{1/\alpha}[1-s_x(1-1/\sigma)][1-\theta(1-s_x)^{\sigma-1}]}{[1-s_n(1-1/\sigma)-\theta(1-s_x)^{\sigma}(1-s_n)](1-s_n)^{1/\alpha}+\Delta^{1/2}} ,
$$

using the expression of $k^{**}$ given in Proposition 1 for substitution.

For $0 \leq s_x \leq 1/\sigma$, showing $|dk_t/dk_{t-1}|_{k_{t-1} = k^{**}} < 1$ is equivalent to showing

$$
F \equiv \{1-s_n(1-1/\sigma)-\theta(1-s_x)^{\sigma}(1-s_n)+(sA/\sigma)(1-s_n)^{1/\alpha}+\Delta^{1/2}\}^2
$$

$$
+4(sA/\sigma)(1-s_n)^{1/\alpha}[1-s_x(1-1/\sigma)][1-\theta(1-s_x)^{\sigma-1}] > 0
$$

whereby $1-\theta(1-s_x)^{\sigma-1} \leq 0$. Using the expression for $\Delta$ in $F$ leads to

$$
F = [1-s_n(1-1/\sigma)-\theta(1-s_x)^{\sigma}(1-s_n)+(sA/\sigma)(1-s_n)^{1/\alpha}]^2 + \Delta + 2\Delta^{1/2}[1-s_n(1-1/\sigma)-\theta(1-s_x)^{\sigma}(1-s_n)+(sA/\sigma)(1-s_n)^{1/\alpha}] + 4(sA/\sigma)(1-s_n)^{1/\alpha}[1-s_n(1-1/\sigma)][1-\theta(1-s_x)^{\sigma-1}] + 2[1-s_n(1-1/\sigma)-\theta(1-s_x)^{\sigma}(1-s_n)]^2 + 2[(sA/\sigma)(1-s_n)^{1/\alpha}]^2 + 4(sA/\sigma)(1-s_n)^{1/\alpha}[\theta(1-s_x)^{\sigma-1}[s_x(1-s_n)+s_n/\sigma]+[1-s_n(1-1/\sigma)]
$$

$$
[1-\theta(1-s_x)^{\sigma-1}]+2\Delta^{1/2}[1-s_n(1-1/\sigma)-\theta(1-s_x)^{\sigma}(1-s_n)+(sA/\sigma)(1-s_n)^{1/\alpha}] + 2[1-s_n(1-1/\sigma)-\theta(1-s_x)^{\sigma}(1-s_n)]^2 + 2[(sA/\sigma)(1-s_n)^{1/\alpha}]^2 + 4(sA/\sigma)(1-s_n)^{1/\alpha}[\theta(1-s_x)^{\sigma-1}[s_x(1-s_n)+s_n/\sigma]+[1-s_n(1-1/\sigma)]
$$

Here, $[\theta(1-s_x)^{\sigma-1}[s_x(1-s_n)+s_n/\sigma]+[1-s_n(1-1/\sigma)][1-\theta(1-s_x)^{\sigma-1}])$ can be shown to be equal to $[1-s_n(1-1/\sigma)-\theta(1-s_x)^{\sigma}(1-s_n)]$. Thus, we can have
\[ F = 2\left[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma (1-s_n)\right]^2 + 4(sA/\sigma)(1-s_n)^{1-\sigma} \left[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma (1-s_n)\right] + 2\Delta^{1/2} \left[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma (1-s_n) + (sA/\sigma)(1-s_n)^{1-\sigma} \right] \]
\[ = 2\left[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma (1-s_n) + (sA/\sigma)(1-s_n)^{1-\sigma} \right]^2 + 2\Delta^{1/2} \left[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma (1-s_n) + (sA/\sigma)(1-s_n)^{1-\sigma} \right]. \]

A sufficient yet unnecessary condition for \( F > 0 \) is
\[ [1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma (1-s_n) + (sA/\sigma)(1-s_n)^{1-\sigma}] > [1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma (1-s_n) + (1-s_x)(1-s_n)] \text{ (under } G > 1). \]
\[ > 0 \]

This condition is satisfied by the stated conditions on the subsidy rates:
\[ s_n > \sigma(\theta - 2)/[1+\sigma(\theta - 2)] \in (0, 1) \text{ under } \theta > 2 \text{ at } s_x = 0; \text{ or } s_x \to 1/\sigma \text{ at } s_n = 0. \]

Namely, if the subsidy rates are sufficiently high for \( 0 \leq s_x \leq 1/\sigma \), then the economy will eventually converge toward the steady state, or the balanced growth path. We depict case (2) in Figures 2 and 3.

For the special case without any subsidization, \( G = G_0 = sA/\sigma \) and we have
\[ \left. \frac{dk_t}{dk_{t-1}} \right|_{s_n=s_x=0} \frac{1-\theta}{G} < 0, \text{ at } s_x = s_n = 0. \]

The absolute value of this slope exceeds one (unstable \( k^{**} \)) if and only if \( 1 < G < \theta - 1 \) as in the original model of Matsuyama (1999). This condition applies under \( \theta > 2 \).

Finally, if \( s_x > 1/\sigma \), showing \( dk_t/dk_{t-1} \mid_{s_n=s_x=0} < 1 \) is equivalent to showing the following,
\[ \left\{1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma (1-s_n) + (sA/\sigma)(1-s_n)^{1-\sigma} + \Delta^{1/2} \right\}^2 - 4(sA/\sigma)(1-s_n)^{1-\sigma} \left[1-s_n(1-1/\sigma)\right][1-\theta(1-s_x)^{\sigma-1}] > 0 \]
whereby \( 1-\theta(1-s_x)^{\sigma-1} > 0 \). The left-hand side of this inequality can be further decomposed into
\[
\left[ 1 - s_n \left( 1 - 1/\sigma \right) - (sA/\sigma)(1-s_n)^{1-1/\sigma} \right]^2 + \left[ \Theta (1-s_x)^{\sigma} (1-s_n)^{1-1/\sigma} \right]^2 + \Delta \\
+ 2\Delta^{1/2} \left[ 1 - s_n \left( 1 - 1/\sigma \right) - \Theta (1-s_x)^{\sigma} (1-s_n) + (sA/\sigma)(1-s_n)^{1-1/\sigma} \right] \\
+ 2\Theta \left[ 1 - s_n \left( 1 - 1/\sigma \right) \right] (1-s_n)^{1-1/\sigma} (1-s_x)^{1-1/\sigma} \left[ sA/\sigma - (1-s_x)(1-s_n)^{1/\sigma} \right] \\
+ 2\Theta \left( sA/\sigma \right) (1-s_x)^{\sigma-1} (1-s_n)^{1-1/\sigma} \left[ s_n/\sigma + s_x (1-s_n) \right]
\]

which is strictly positive under \( G > 1, \quad 0 \leq s_n < 1 \) and \( 1/\sigma < s_x < 1 \). We illustrate case (3) in Figure 4. Q.E.D.

According to Proposition 2, both types of subsidies can eventually eliminate cycles once their rates are set high enough such that \( |dk_i / dk_{i-1}| < 1 \) under which the balanced growth path with innovation becomes stable. This is achieved either through strengthening the demand for new intermediates (via a higher \( s_x \)) or through reducing the innovation cost (via a higher \( s_n \)) such that R&D activities are profitable even at a low capital-variety ratio. By increasing varieties, the subsidization can exert different impacts on final output and thus on capital investment with a constant saving rate. First, it can directly increase final output by increasing the number of varieties according to equation (11). Second, it can indirectly reduce final output by reducing the equilibrium quantity of each type of intermediate input, as the subsequent increase in the total fixed innovation cost competes for the given amount of initial capital, except that a higher \( s_x \) increases the equilibrium quantity of each new intermediate for \( 0 \leq s_x < 1/\sigma \).

It is convenient to look at how these effects work for the stability of the dynamic system by using a general expression for the slope of the transition curve \( k_i = \Phi(k_{i-1}) \):

\[
dk_i / dk_{i-1} = d(K_i / N_i) / dk_{i-1} = [N_i (dK_i / dk_{i-1}) - K_i (dN_i / dk_{i-1})] / N_i^2.
\]

Rewrite it as the following ways that may help our interpretation:
The sign of \( \frac{dk_t}{dk_{t-1}} \) is determined by the terms in the bracket on the right-hand side. It is positive (negative) if the ratio of the derivative of capital investment to the derivative of variety expansion with respect to the initial abundance of capital, \( \frac{(dK_t / dk_{t-1})}{(dN_t / dk_{t-1})} \), is greater (smaller) than the resultant capital-variety ratio, \( k_i \). The absolute value of \( \frac{dk_t}{dk_{t-1}} \) depends positively on the difference between the two responses, as fractions of their new stocks, \( \frac{((dK_t / dk_{t-1}) / K_t) - (dN_t / dk_{t-1}) / N_t}{N_t} \), as well as on the new capital-variety ratio, \( k_i / N_t \). In the steady state with \( k_{t-1} = k_i > k_c \), both the sign and the magnitude of \( \frac{dk_t}{dk_{t-1}} \) will depend solely on the gap in the respective elasticity of \( K_t \) and \( N_t \) with respect to \( k_{t-1} \).

Define \( G_0 = sA / \sigma \) that serves as the growth factor in the absence of subsidization. If \( k_{t-1} > k_c \), from (14) and (19a), \( \frac{dk_t}{dk_{t-1}} = G_0 (1 - s_n)^{1/\sigma} N_{t-1} / [1 - s_n (1 - 1/\sigma)] > 0 \) holding \( N_{t-1} \) constant, while from (19b), \( \frac{dN_t}{dk_{t-1}} = \theta N_{t-1} / [1 - s_n (1 - 1/\sigma)] > 0 \). Here, both capital investment and variety expansion respond positively to the initial abundance of capital per variety.

Under the assumption \( 1 < G < \theta - 1 \), however, in the absence of subsidization the variety response to the initial abundance of capital is more sensitive than the investment response, causing instability of the balanced growth path in the original Matsuyama model. Without subsidization, in the steady state or on the balanced growth path,
Proposition 1 and equation (19b) lead to

\[ k^{**} = \frac{G_0 - 1 + \theta}{\theta} > 1 \text{ for } G_0 > 1 \text{ and } s_i = s_u = 0; \ N_i / N_{t-1} = G_0. \]

So \( dk_i / dk_{i-1} = [(dK_i / dk_{i-1}) - k_i (dN_i / dk_{i-1})] / N_i \) at the steady state \( k^{**} \) on the balanced growth path equals \( dk_i / dk_{i-1} \big|_{k_{i-1} = k} = (G_0 - k^{**} \theta) N_{t-1} / N_i = (1 - \theta) / G_0 \), which is negative under \( 1 < \theta \) and smaller than \(-1\) under \( 2 < \theta \). In this case, period-two cycles prevail and persist forever.

Subsidizing the fixed innovation cost strengthens the response of variety expansion to the initial abundance of capital, \( dN_i / dk_{i-1} = \theta N_{t-1} / [1 - s_n (1 - 1/\sigma)] > 0 \), but weakens the response of investment, \( dK_i / dk_{i-1} = G_0 (1 - s_n)^{1/\sigma} N_{t-1} / [1 - s_n (1 - 1/\sigma)] > 0 \), when setting \( s_x = 0 \). From (22), if this subsidy is large enough, at least for \( s_n > 1/(2 - 1/\sigma) \in (0, 1) \), a further increase in \( s_n \) will also lead to lower capital per variety \( k_i \) as long as \( k_i > k_c \) and \( k_{i-1} > k_c \), because beyond the level \( s_n = 1/(2 - 1/\sigma) \in (0, 1) \) the numerator of \( k_i \) starts to decrease with \( s_n \). Combining them together, the sign of \( dk_i / dk_{i-1} = [G_0 (1 - s_n)^{1/\sigma} - k_i \theta] (N_{t-1} / N_i) / [1 - s_n (1 - 1/\sigma)] \) is only determined by the factor \( [G_0 (1 - s_n)^{1/\sigma} - k_i \theta] \) whereby both terms, \( G_0 (1 - s_n)^{1/\sigma} \) and \( k_i \theta \), eventually decline with the subsidy rate on the innovation cost when the subsidy rate becomes large enough. This helps to explain why the sign of \( dk_i / dk_{i-1} \) remains negative for all levels of the subsidy rate in the regime with innovation. The remaining factors that only determine the magnitude, not the sign, of \( dk_i / dk_{i-1} \) are decreasing with \( s_n \) as well:

\[ (N_{t-1} / N_i) / [1 - s_n (1 - 1/\sigma)] = [1 - s_n (1 - 1/\sigma) + \theta [k_{i-1} - (1 - s_n)]]^{-1} \text{ for } \theta > 1. \]

This explains why the absolute value of \( dk_i / dk_{i-1} \) becomes smaller when the subsidy rate \( s_n \) becomes larger.
Subsidizing the purchase of new intermediates does not affect the responses of capital investment and variety expansion to the initial abundance of capital, when setting \( s_n = 0 \). The sign of \( \frac{dk_i}{dk_{i-1}} \) is merely determined by \([G_0 - k\theta]\) which is initially negative when the subsidy rate is equal to zero. It follows from (22) that a higher subsidy rate for the purchase of new intermediates will reduce the amount of capital per variety \( k_t \) as long as \( k_t > k_c \) and \( k_{t-1} > k_c \), because \( \frac{dk_i}{ds_j} \) is signed by \([1 - \sigma s_x - \theta (1 - s_x)\sigma - \theta k_{t-1} (\sigma - 1)]\) in which \( 1 - \sigma s_x - \theta (1 - s_x)\sigma \) attains a negative maximum at \( s_x = 1/\sigma \). Consequently, a higher subsidy rate for the purchase of new intermediates will reduce the absolute value of \([G_0 - k\theta]\). At \( G_0 = k^{\pi} \theta \) on the balanced growth path, the solution for this subsidy rate is \( s_x = 1/\sigma \) that leads to \( \frac{dk_i}{dk_{i-1}} \bigg|_{k_t = k^{\pi}} = 0 \). When the subsidy rate is increased further for \( s_x > 1/\sigma \), \([G_0 - k^{\pi} \theta] > 0 \) must hold, leading to \( \frac{dk_i}{dk_{i-1}} > 0 \) on the balanced growth path. Recall that subsidizing the purchase of new intermediates at a rate \( s_x > 1/\sigma \) will lead to \( x_s < x_m \), thereby creating a loss in final output. The loss in final output, due to a higher \( s_x \) beyond \( s_x = 1/\sigma \) will in turn lead to a decline in capital investment for a constant saving rate given any initial \( k_{t-1} \), while variety expansion accelerates at a higher \( s_x \). Consequently, a higher \( s_x \) with \( s_x > 1/\sigma \) will reduce \( k^{\pi} \) to a level such that \([G_0 - k^{\pi} \theta] > 0 \) and thus \( \frac{dk_i}{dk_{i-1}} > 0 \) at the steady state. For \( s_n = 0 \), the absolute value of \( \frac{dk_i}{dk_{i-1}} \bigg|_{k_t = k^{\pi}} \) on the balanced growth path is derived below:

\[
\frac{dk_i}{dk_{i-1}} \bigg|_{k_t = k^{\pi}} = \frac{G_0 + 1 - \theta (1 - s_x)\sigma - \Delta^{1/2}}{G_0 + 1 - \theta (1 - s_x)\sigma + \Delta^{1/2}} < 1.
\]

Therefore, the balanced growth path becomes stable once the subsidy rate on the purchase of new intermediates is set high enough such that \( \bigg| \frac{dk_i}{dk_{i-1}} \bigg|_{k_t = k^{\pi}} < 1 \).
4. Numerical simulation of welfare comparison and optimal subsidy rates

Since the two kinds of subsidies can promote growth and innovation and can eliminate cyclical fluctuations, they have potential for enhancing social welfare compared with the benchmark model without any government subsidization. Starting with lower final output and slower innovation and growth under monopoly pricing compared to their socially optimal levels in the Romer regime (as shown in Barro and Sala-i-Martin, 1995), increasing the subsidies may have different impacts on final output on the one hand and on innovation and growth on the other. Thus, increasing the subsidy rates can have opposing impacts on welfare. The positive effect of subsidies on innovation and growth tends to enhance welfare when the innovation rate and the growth rate are lower than their socially optimal levels. The effect of subsidies on final output is initially positive at low subsidy rates and eventually negative at sufficiently high subsidy rates, as shown earlier in our model. Moreover, when subsidies mitigate cyclical fluctuations in consumption, investment and innovation, there are possible efficiency gains due to diminishing marginal utility and diminishing marginal product. Thus, the overall welfare effect is expected to be initially positive, when the subsidy rates are low, but eventually negative, when the subsidy rates become high enough. Unlike existing studies of R&D subsidization that focus on the balanced growth path only, the model here allows us to explore whether the optimal rates of subsidies lie in the range where they eventually eliminate the cycles with oscillatory or monotonic convergence toward the steady state on the balanced growth path.

In this section, we will gauge the welfare gains from the subsidies and find optimal subsidy rates. We will do so numerically, given the complexity of carrying it out analytically.
To this end, we assume a far-sighted government that has a social welfare function as:

$$U^S = \sum_{t=1}^{\infty} \beta^{t-1} U^t, \quad 0 < \beta < 1,$$

(23)

where $\beta^{t-1}$ is the discount factor.

We set a benchmark parameterization: $s = 0.4, A = 15, \beta = 0.6, \theta = 2.4414,$ and $\sigma = 5.$ The value of $\sigma = 5$ is in line with that in Matsuyama (1999) where it plays dual roles: $1 - 1/\sigma = 0.8$ is the share of capital (interpreted broadly as both physical and human capital); $1/(\sigma - 1) = 0.25$ is the monopoly mark-up enjoyed by the innovator. Taking one period as 30 years, the value of the discounting factor per period at $\beta = 0.6$ corresponds to an annual discounting factor of 0.9855 as in Gomme et al. (2001). It is also plausible to set the taste parameter for young-age consumption as 0.6 compared with that for old-age consumption as 0.4. The resultant equilibrium saving rate is 0.4.

According to Proposition 2, the benchmark parameterization without subsidies, $s_t = 0$ and $s_n = 0,$ satisfies $1 < G < \theta - 1,$ indicating that the economy grows through period-2 cycles, perpetually moving back and forth between the two regions. This is an empirically plausible case as argued in Matsuyama (1999). Interesting questions in this numerical example are as follows: How can the subsidies affect the dynamics of this economy? Are there any social welfare gains from the subsidization? What are the possible optimal rates of these two kinds of subsidies?

To answer these questions, we choose the initial states as $K_0 = 0.4$ and $N_0 = 1$ and take 1000 periods or generations into account to determine the social welfare approximately as in (23), using the first-order difference equations in (19a) and (19b) to update the states $K_t$ and $N_t$ and then finding final output, consumption, the tax rate and utility in every
period. In Table 1 and Table 2, we report the simulation results when increasing $s_s$ and $s_n$ from zero to reach a peak of the welfare level, one at a time, respectively.

In Table 1, we set $s_n = 0$ and examine the dynamic behavior, the balanced growth rate, the consumption tax rate, the variety of intermediates, and the welfare level, when varying $s_s$ from 0 to 40%. When $s_s$ is sufficiently small (e.g., $s_s = 0.01, s_s = 0.02$) the economy still alternates between the policy-dormant and the policy-active regions, and the social welfare is higher than the case without subsidization. When $s_s$ is increased further but still below $1 / \sigma = 0.2$, the steady state $k^{**}$ becomes stable and the economy achieves oscillatory convergence toward it as in case 2 of Proposition 2. As $s_s$ is increased beyond $1 / \sigma = 0.2$, the condition for case 3 in Proposition 2 is satisfied for $k_s$ to converge toward $k^{**}$ monotonically. For a better view of the welfare effect, in Figure 5 we vary $s_s$ from 0 to 0.8 ($s_n = 0$) and find the subsequent welfare levels. It is worth noting that the welfare level is a concave and smooth curve, peaking at a unique point when $s_s = 0.22$, due to a balance between the various gains and losses in efficiency mentioned earlier.

Note that this optimal subsidy rate exceeds $1 / \sigma = 0.2$, at which it maximizes the static efficiency by equalizing the user prices of old and new intermediates. Below $1 / \sigma = 0.2$, a higher subsidy rate $s_s$ yields not only a gain in static efficiency by closing down the price gap for new and old intermediates but also a gain in dynamic efficiency by promoting variety expansion. Beyond $1 / \sigma = 0.2$, a further increase in $s_s$ will widen the price gap and cause a loss in static efficiency because now $s_s > 1 / \sigma$. However, increasing the subsidy rate beyond $1 / \sigma = 0.2$ engenders monotonic convergence toward the balanced growth path with innovation, thereby creating a welfare gain via consumption smoothing. The resultant
welfare level at the optimal subsidy rate is 3.0135, significantly higher than that without subsidization, 2.6727. The welfare gain here is in terms of log utility; it will be measured again by an equivalent consumption variation later.

In Table 2, we fix $s_x = 0$ and focus on the effects of the subsidy on the fixed R&D cost, $s_n$. Period-2 cycles persist for a relatively wide range of $s_n$. After that, when it is high enough to satisfy the condition for case 2 in Proposition 2, period-2 cycles are replaced by oscillatory convergence to $k^{*}$ in the policy-active region. In Figure 6, we vary $s_n$ from 0 to 90% ($s_x = 0$) and find the subsequent welfare level. It is worth noting that the welfare level is a concave and smooth curve as well, peaking at a unique point when $s_n^* = 0.70$ that achieves oscillatory convergence toward the steady state with balanced growth in capital and variety. This welfare curve is flatter before peaking and takes longer to reach the optimal level of the subsidy rate than in Figure 5, because this subsidy $s_n$ does not change the price gap and therefore does not create the additional gain or loss in static efficiency as those created by $s_x$ on either side of $s_x = 1/\sigma$. So the efficiency gain from faster variety expansion and from convergence to the steady state at a higher subsidy rate to the fixed R&D cost is gradually offset by a loss from the subsequent decline in the equilibrium quantity of each intermediate. Beyond this optimal subsidy rate, the welfare level declines rapidly, because the efficiency loss from the declined use of each intermediate is increasing at the margin. The resultant welfare level is 3.028, significantly higher than that without subsidization, 2.6727.

Finally, for a better gauge of the welfare gains, we report their equivalent consumption variations in the last column of the tables in terms of the percentage change in consumption,
denoted $\Delta$, for each generation such that the benchmark case without subsidization reaches the same welfare level as that in a case with subsidization type $i$:

$$U_{\text{no-subsidization}}^s + \frac{1}{1-\beta} \ln(1+\Delta) = U_{s_i}^s.$$  

This corresponds to adding $\sum_{t=0}^{\infty} \beta^t \ln(1+\Delta) = \left[ \ln(1+\Delta) \right] / (1-\beta)$ to the welfare level in the benchmark case without subsidization. According to the reported figures, the optimal subsidy to the purchase of new intermediates and the optimal R&D investment subsidy can achieve welfare gains equivalent to a rise in consumption for each generation of 14.60% and 15.27% respectively. Because the respective optimal rates of subsidies mitigate and eventually eliminate the cycles, part of the welfare gain must have come from smoothing consumption, which is new compared to the results in existing analysis of subsidization on the balanced growth path alone.

5. Conclusion

We have examined the implications of two types of subsidies, one type to the purchase of new intermediate products and the other to R&D investment, in the Matsuyama model of growth through endogenous cycles (1999). One contribution of doing so is that the subsidization can reduce the critical level of the capital-variety ratio substantially, enhancing the possibility for the economy to stay at the policy-active region with sustainable innovation and growth. Sufficient subsidization can rule out the neoclassical regime without innovation from the steady state in the long run.

Another contribution is that we have characterized several possible scenarios of the asymptotic paths of the overlapping generation economy from any initial state depending on
the values of the economic fundamentals and subsidy rates. In the most interesting yet empirically plausible scenario, sufficient subsidization leads to a stable balanced growth path with innovation.

Also, we have used a numerical example to gauge the welfare gains from the two types of subsidies based on an empirically plausible parameterization for a period-2 cycle economy in the absence of subsidization. It turns out that both types of subsidies can help enhance the welfare level significantly in terms of a 14-15% rise in consumption for each generation; and the optimal subsidy rates maximizing social welfare are calculated in their plausible ranges that eventually eliminate cycles for consumption smoothing.

Our results in this paper appear consistent not only with substantial subsidization to new investment and R&D spending but also with the combination of intensified innovation and dampened cyclical fluctuations in some industrial nations in the postwar era.

The essence of our results may also apply to the different model of growth through cycles in Francois and Shi (1999) where labor is the sole input through multiple periods for innovation success and for intermediate goods production. They identify the aggregate income externality, in the form of spillovers from innovators' temporary profits to aggregate income and back to innovators profits, as the reason for innovation cycles. R&D subsidization aiming to enhance the profitability of innovation may therefore change the length of cycles and help internalize the spillover for welfare gains. A full analysis of subsidization in such an alternative setup combining the neo-Keynesian externality with the neo-Schumpeterian innovation awaits future research.
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Table 1. Results of changing the subsidy to the purchase of new intermediate goods

Benchmark parameters: \( s = 0.4, A=15, \beta=0.6, \theta=2.4414, \sigma=5, K_0 = 0.4, N_0 = 1, s_n = 0 \)

<table>
<thead>
<tr>
<th>Subsidy rates</th>
<th>Mode of convergence</th>
<th>Steady state ((k^L, k^H))</th>
<th>Growth rate (annual rate%)</th>
<th>Tax rate (%)</th>
<th>Product variety (N_{1000})</th>
<th>Welfare</th>
<th>(\Delta)</th>
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<tr>
<td>( s_s = 0 )</td>
<td>Period-2 cycle</td>
<td>(0.9817, 1.1824)</td>
<td>1.2022 (0.62)</td>
<td>0</td>
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<td>( s_s = 0.01 )</td>
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<td>( s_s = 0.02 )</td>
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<td>1.2223 (0.67)</td>
<td>(0, 0.32)</td>
<td>7.2916*10^86</td>
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<td>( s_s = 0.05 )</td>
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<td>0.8785</td>
<td>1.2556 (0.76)</td>
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<td>4.0717*10^98</td>
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<td>( s_s = 0.10 )</td>
<td>Oscillatory</td>
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<td>( s_s = 0.15 )</td>
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<td>1.359 (1.03)</td>
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<td>( s_s = 0.22 )</td>
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<td>( s_s = 0.25 )</td>
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<td>( s_s = 0.30 )</td>
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<td>5.6794*10^158</td>
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Note: (1) The growth rate in the period-2 cycle economy is calculated as the geometric average of the corresponding growth rates in the two regions following Matsuyama (1999).

(2) The values in the brackets beside growth rates indicate the discounted annual rates.

(3) The subsidy rates with * indicates the optimal rate maximizing the social welfare.
Table 2. Results of changing the subsidy on the fixed R&D cost

Benchmark parameters:  \( s = 0.4, A=15, \beta=0.6, \theta = 2.4414, \sigma = 5, K_0 = 0.4, N_0 = 1, s_x = 0 \)

<table>
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<tr>
<th>Subsidy rates</th>
<th>Mode of convergence</th>
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<th>Tax rate (%)</th>
<th>Product variety ( N_{1000} )</th>
<th>Welfare ( U^S ) (%)</th>
<th>( \Delta )</th>
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<tr>
<td>0</td>
<td>Period-2 cycle</td>
<td>(0.9817, 1.1824)</td>
<td>1.2022 (0.62)</td>
<td>0</td>
<td>4.5801*10^79</td>
<td>2.6727</td>
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<td>0.10</td>
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<td>(0, 0.33)</td>
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Note: (1) The growth rate in the period-2 cycle economy is calculated as the geometric average of the corresponding growth rates in the two regions following Matsuyama (1999).
(2) The values in the brackets beside growth rates indicate the discounted annual rates.
(3) The subsidy rates with * indicates the optimal rate maximizing the social welfare.
Figure 1. \( G < 1 \)

Figure 2. \( G > 1, s_x < 1/\sigma \) and \( \left| \frac{dk}{d\gamma_{t-1}} \right|_{k_{t-1} = k^*} > 1 \)
Figure 3. $G > 1$, $s_x < 1/\sigma$ and $|dk_v/dk_{v-1}|_{k_{v-1}=k''} < 1$

Figure 4. $G > 1$ and $s_x > 1/\sigma$
Figure 5. Welfare and the subsidy to intermediate goods

Figure 6. Welfare and the subsidy to R&D investment