Social choice of convex risk measures through Arrovian aggregation of variational preferences

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Proof idea
This paper studies the social choice problem for variational preferences. Impossibility results (necessity of dictatorship) and possibility results are obtained. Variational preferences are represented by minimax expected utility functions with additive penalty; thus the findings apply to aggregation of (negated) convex risk measures as well. The proof uses a model-theoretic (mathematico-logical) approach to aggregation theory: Arrow-rational aggregators are written as ultraproduct constructions with respect to the ultrafilter of decisive coalitions.
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Framework

- *Individuals* have variational preferences over lotteries; these are represented by minimax expected utility functions with additive lower-semicontinuous penalty (Maccheroni-Marinacci-Rustichini 2006) and hence by negated convex risk measures (Föllmer-Schied 2004).
- The individual preferences are aggregated into a *social preference relation* through a social choice function (aggregator).
- *Rationality axioms for the aggregator* inspired by Arrow (1963) are imposed.
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Motivating questions:

- Under which rationality axioms will the *social preference relation* be *variational* (corresponding to a convex risk measure)?
- When will the *aggregator be non-dictatorial*?

Results:

- Possibility results: Fishburn’s (1970) theorem
- Impossibility results: Analogue’s of Arrow’s (1963) and Campbell’s (1990) theorems.
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The proof is based on three key steps:

- translating variational preference aggregation into an abstract framework of relational-structure aggregation;
- invoking a one-to-one correspondence between Arrovian aggregators and ultrafilters (generalizing Kirman-Sonnermann 1972) via the set of decisive coalitions;
- representing of Arrovian aggregators as restricted ultraproduct constructions (generalizing Lauwers-Van Liedekerke 1995).

The proofs of the (im)possibility theorems then follow easily from Łoś's theorem.
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Proof idea
States, consequences, acts

Consider:
- a finite set $S$ (set of states of the world),
- a convex subset $X$ of a vector space $Y$ with more than one element (set of consequences),

Define:
- $\mathcal{F} = X^S$ (the set of simple acts), a convex subset of the vector space $Y^S$
- $\mathcal{F}_c \subseteq \mathcal{F}$ (the set of constant simple acts), the set of all constant functions from $S$ to $X$.

Variational preference relations are binary relations $\succeq$ among acts, i.e. subsets of $\mathcal{F} \times \mathcal{F}$

The symmetric part $\sim$ of $\succeq$ is given by
\[
\forall f, g \in \mathcal{F} \quad f \sim g \iff (f \succeq g, \ g \succeq f).
\]

The asymmetric part $\succ$ of $\succeq$ is given by
\[
\forall f, g \in \mathcal{F} \quad f \succ g \iff (f \succeq g, \ g \not\succeq f).
\]
Definition of variational preferences

A binary relation $\succsim$ on $\mathcal{F}$ with symmetric part $\sim$ and asymmetric part $\succ$ is a variational preference ordering or convex risk-preference ordering if and only if it satisfies all of the following axioms:

(A1) Weak order properties. For all $f, g \in \mathcal{F}$, either $f \succsim g$ or $g \succsim f$ (completeness); for all $f, g, h \in \mathcal{F}$, if $f \succsim g$ and $g \succsim h$, then $f \succsim h$ (transitivity).

(A2) Weak certainty independence. For all $f, g \in \mathcal{F}, x, y \in \mathcal{F}_c$ and $\alpha \in (0, 1)$,

$$\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x \Rightarrow \alpha f + (1-\alpha)y \succsim \alpha g + (1-\alpha)y.$$

(A3) Continuity. For all $f, g, h \in \mathcal{F}$, the sets

$$\{\beta \in [0, 1] : \beta f + (1-\beta)g \succsim h\}$$

and

$$\{\beta \in [0, 1] : h \succsim \beta f + (1-\beta)g\}$$

are closed.

(A4) Monotonicity. For all $f, g \in \mathcal{F}$, if $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.

(A5) Uncertainty aversion. For all $f, g \in \mathcal{F}$ and $\alpha \in (0, 1)$, if $f \sim g$, then $\alpha f + (1-\alpha)g \succsim f$.

(A6) Non-degeneracy. There exist $f, g \in \mathcal{F}$ such that $f \succsim g$. 
Alternative characterization of continuity

Definition
Let $f, g, h \in \mathcal{F}$ and $\beta \in [0, 1]$. A pair of real numbers $(\alpha, \gamma) \in [0, 1]^2$ is called a witness-pair to the continuity of $\succeq$ along $f, g, h \in \mathcal{F}$ in $\beta$ if and only if for all $\delta \in (\alpha, \gamma)$, one has
1. $\delta f + (1 - \delta) g \prec h$ if $\beta f + (1 - \beta) g \prec h$ and
2. $h \prec \delta f + (1 - \delta) g$ if $h \prec \beta f + (1 - \beta) g$,
whilst either $\alpha < \beta < \gamma$ or $0 = \alpha = \beta < \gamma$ or $\alpha < \beta = \gamma = 1$.

A real number $\varepsilon \in [0, 1]$ is called a witness to the continuity of $\succeq$ along $f, g, h \in \mathcal{F}$ in $\beta$ if and only if there exists some $\alpha \in [0, 1]$ or $\gamma \in [0, 1]$ such that either $(\alpha, \varepsilon)$ or $(\gamma, \varepsilon)$ is a witness-pair to the continuity of $\succeq$ along $f, g, h \in \mathcal{F}$ in $\beta$.

Remark
If $\succeq$ satisfies completeness (A1a), then $\succeq$ satisfies continuity (A3) if and only if for all $f, g, h \in \mathcal{F}$ and all $\beta \in [0, 1]$ there exists a witness to the continuity of $\succeq$ along $f, g, h \in \mathcal{F}$ in $\beta$. 
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Proof idea
Consider a finite or infinite set $N$ (the *population set* or *electorate*). Elements of $N$ are called *individuals*, subsets of $N$ are called *coalitions*. Suppose that each individual $i \in N$ is endowed with a variational preference ordering $\succsim_i$; any such resulting $N$-sequence $\succsim = (\succsim_i)_{i \in N}$ is called a *variational preference profile*.
Uniform continuity and discontinuity properties for profiles

Some of the (im)possibility results will depend on how much the profiles in the population differ in their continuity. A profile $\succsim$ is said to be *continuous* if and only if $\succsim_i$ is continuous for all $i \in \mathbb{N}$.

A profile $\succsim$ is *discontinuous in the limit* if and only if for all $f, g, h \in \mathcal{F}$ and all $\beta \in [0, 1]$, every $\alpha \in [0, 1]$ is a witness to the continuity of $\succsim_i$ along $f, g, h$ in $\beta$ for only finitely many $i \in \mathbb{N}$.

A profile $\left(\succsim_i\right)_{i \in \mathbb{N}}$ is said to be *uniformly continuous* if and only if for all $f, g, h \in \mathcal{F}$ and all $\beta \in [0, 1]$, there exist $\alpha, \gamma \in [0, 1]$ which for all $i \in \mathbb{N}$ are a witness-pair to the continuity of $\succsim_i$ along $f, g, h$ in $\beta$. 
Denote by $\mathcal{P}$ the set of all variational preference relations on $\mathcal{F}$. A preference aggregator is a map $F$ with domain $\subseteq \mathcal{P}^N$ and range $\subseteq \mathcal{F} \times \mathcal{F}$ such that only complete binary relations are in its range. A variational preference aggregator is a preference aggregator with range $\subseteq \mathcal{P}$. 
A preference aggregator $F$ is

- *universal* if and only if $\text{dom}(F) = \mathcal{P}^N$ (so that $F : \mathcal{P}^N \to \mathcal{P}$);
- *weakly universal* if and only if $\text{dom}(F)$ is a *rich aggregator domain*. Herein, a subset $\mathbb{D} \subseteq \mathcal{P}^N$ is called a *rich aggregator domain* if and only if there are $f, f', g, g' \in \mathcal{F}$ and variational preference orderings $\succeq_1, \succeq_2, \succeq_3$ such that
  
  - $f \succeq_1 g$, $f' \succeq_1 g'$, $f \succeq_2 g$, $f' \prec_2 g'$, $f \prec_3 g$, $f' \succeq_3 g'$, and
  
  - $\{\succeq_1, \succeq_2, \succeq_3\}^N \subseteq \mathbb{D}$;

**Remark**

*If $S$ contains at least two elements, then $\mathcal{P}^N$ is a rich aggregator domain, and every universal aggregator is also weakly universal.*
Other rationality properties of aggregators

A preference aggregator $F$ is

- **systematic** if and only if for every $\succeq \in \text{dom}(F)$ and all $f, f', g, g' \in \mathcal{F}$ with \( \{ i \in N : f \succeq_i g \} = \{ i \in N : f' \succeq_i g' \} \) one has
  \[
  f \ F \left( \succeq \right) g \iff f' \ F \left( \succeq \right) g';
  \]

- **Paretian** if and only if for every $\succeq \in \text{dom}(F)$ and all $f, g \in \mathcal{F}$, if $f \succeq_i g$ for all $i \in N$, then $f \ F \left( \succeq \right) g$;

- **dictatorial** if and only if there exists some $i \in N$ (called dictator) such that for every $\succeq \in \text{dom}(F)$ and all $f, g \in \mathcal{F}$,
  \[
  f \ F \left( \succeq \right) g \iff f \succeq_i g.
  \]
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Theorem
Let $N$ be finite and let $F$ be a (variational) preference aggregator. $F$ is weakly universal, systematic and Paretian if and only if it is dictatorial.
Theorem
Let $N$ be an arbitrary set (finite or infinite). Let $F$ be a weakly universal, systematic and Paretian variational preference aggregator. Suppose that its domain $\text{dom}(F)$ contains a profile $\succ$ that is (continuous, but) discontinuous in the limit. Then $F$ is dictatorial.
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Fishburn’s theorem for variational preference aggregation

Theorem
Let $N$ be an infinite set, and let $\mathbb{D} \subseteq \mathcal{P}^N$ be a rich aggregator domain such that all profiles in $\mathbb{D}$ are uniformly continuous. Then there exist non-dictatorial, weakly universal, systematic and Paretian variational preference aggregators $F : \mathbb{D} \rightarrow \mathcal{P}$. 
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Relational structures for variational preferences

One can view variational preferences and all their structure as given by relations on $\mathcal{F}$:

1. Embed the unit interval into $\mathcal{F}$: Choose two distinct elements $x_0, x_1 \in X$, and define for all $\alpha \in [0, 1]$ a constant function $\vec{\alpha} \in \mathcal{F}_c$ by $\vec{\alpha} : s \mapsto \alpha x_0 + (1 - \alpha) x_1$. Define $\bar{I} = \{ \vec{\alpha} : \alpha \in [0, 1] \}$ and $I = \{ \vec{\alpha} : \alpha \in (0, 1) \} = \bar{I} \setminus \{ x_0, x_1 \}$.

2. Define a mixture operator $m : \bar{I} \times \mathcal{F}^2 \to \mathcal{F}$: For all $\alpha \in [0, 1]$ and $f, g \in \mathcal{F}$, put

   \[ m(\vec{\alpha}; f, g) = \alpha f + (1 - \alpha) g \in \mathcal{F}. \]

3. Define a linear ordering $\prec_{\bar{I}}$ on $\bar{I}$ by

   \[ \vec{\alpha} \prec_{\bar{I}} \vec{\beta} \iff \alpha < \beta \]

   for all $\alpha, \beta \in [0, 1]$. 
Language of variational preferences and their aggregates

Then, the axioms defining a variational preference relation can be captured in a language $\mathcal{L}$ of first-order logic with:

- two unary predicate symbols $\dot{C}$, $\dot{I}$ (for membership in the subsets $F_c$ and $I$, respectively, of $F$),
- two binary predicate symbols $\dot{\bowtie}$ and $\dot{<}$,
- $\text{card}(S)$ operator symbols $\dot{\pi}_s$ (for evaluation of an act at a state $s$),
- and a ternary operation symbol $\dot{m}$ (the mixture operator).

Denote by $\Gamma$ the set of relational structures of the form $(\mathcal{F}, C, I, \bowtie, <\bar{I}, (\pi(s))_{s \in S}, m)$ where all relations except $\bowtie$ have been chosen canonically.

An aggregator can be viewed as a map $F : \Gamma^N \to N$. 
Having described variational preferences through $\mathcal{L}$-formulae, one can define *decisive* coalitions:

A coalition $D$ is said to be *decisive* with respect to $F$ if and only if $D = \{i \in N : (\mathcal{F}, \succsim_i) \models \phi\}$ for some profile $\succsim \in \text{dom}(F)$ and some quantifier-free $\mathcal{L}$-formula $\phi$ such that $\left(\mathcal{F}, F(\succsim)\right) \models \phi$. 
Ultrafilters

A set of coalitions $\mathcal{C}$ is a filter on $N$ if and only if

- $\mathcal{C}$ is non-trivial $\emptyset \neq \mathcal{C} \neq 2^N$;
- $\mathcal{C}$ is closed under intersections ($\forall A, B \in \mathcal{C} \quad A \cap B \in \mathcal{C}$);
- $\mathcal{C}$ is closed under supersets

($\forall A \in \mathcal{C} \forall B \subseteq N \quad (B \supseteq A \Rightarrow B \in \mathcal{C})$)

The set of filters on $N$ is partially order by inclusion. Maximal filters are called ultrafilters.

Ultrafilters on $N$ are in a one-to-one correspondence with $\{0, 1\}$-valued finitely additive measures on $N$: The bijection is given by $\mathcal{U} \mapsto \chi_{\mathcal{U}} (= \mu)$, its inverse by $\mu \mapsto \mu^{-1}\{1\}$ (= $\mathcal{U}$).

Lemma

If $F$ is a weakly universal, systematic and Paretian variational aggregator, then $\mathcal{D}_F$ is an ultrafilter.

For finite $N$, any ultrafilter is the set of supersets of some singleton — for decisive coalitions, this corresponds to dictatorship.
Given a sequence \((A_i)_{i \in \mathbb{N}} \in \Gamma^N\) and an ultrafilter \(U\) on \(N\), one can define a new structure \(\prod_{i \in \mathbb{N}} A_i / U\), the ultraproduct of \((A_i)_{i \in \mathbb{N}}\) through

\[
\prod_{i \in \mathbb{N}} A_i / U \models \phi \iff \{ i \in \mathbb{N} : A_i \models \phi \} \in U \tag{1}
\]

for all quantifier-free \(\phi\). Łoś’s theorem asserts that Equation (1) holds then even for all \(\phi\). For
The ultraproduct’s domain is \(\mathcal{F}^N / U\), which is strictly larger than \(\mathcal{F}\). If \(U = D_F\), then the restriction of the ultraproduct to \(\mathcal{F}\) is just \(F\):

\[
\forall A \in \Gamma^N \quad F(A) = \text{res}_\mathcal{F} \prod_{i \in \mathbb{N}} A_i / D_F.
\]

Combining this with Łoś’s theorem, one has for all universal \(\phi\),

\[
F(A) \models \phi \iff \{ i \in \mathbb{N} : A_i \models \phi \} \text{ decisive.}
\]
Thank you!
K.J. Arrow.
*Social choice and individual values. 2nd ed.*

D.E. Campbell
Intergenerational social choice without the Pareto principle.

P.C. Fishburn
Arrow’s impossibility theorem: Concise proof and infinite voters.

F. Herzberg.
Social choice of convex risk measures through Arrovian aggregation of variational preferences.

F. Herzberg.
Judgment aggregators and Boolean algebra homomorphisms.

F. Herzberg and D. Eckert.
General aggregation problems and social structure: A model-theoretic generalisation of the Kirman-Sondermann correspondence.

A.P. Kirman and D. Sondermann.
Arrow’s theorem, many agents, and invisible dictators.

L. Lauwers and L. Van Liedekerke.
Ultraproducts and aggregation.