A Constructive Study of Markov Equilibria in Stochastic Games with Strategic Complementarities∗

Łukasz Balbus†  Kevin Reffett‡  Łukasz Woźny§

June 2010
(Preliminary Draft)

Abstract

We study a class of discounted infinite horizon stochastic games with strategic complementarities. Using monotone operators on the space of values and strategies, we prove existence of a Stationary Markov Nash equilibrium under different set of assumptions than Curtat (1996), Amir (2002, 2005) or Nowak (2007) via constructive methods. In addition, we provide monotone comparative statics results for ordered perturbations of the space of stochastic games. Under slightly stronger assumptions, we prove the stationary Markov Nash equilibrium values form a complete lattice of Lipschitz stochastic processes, with least and greatest equilibrium value functions being the uniform limit of successive approximations iterations from pointwise lower and upper bounds. Finally, we discuss the relationship between our function-based monotone methods (in pointwise orders), and correspondence-based monotone methods (in set inclusion orders) that have been proposed in the recent literature stemming from the seminal work of Kydland and Prescott (1977, 1980) and Abreu, Pearce, and Stacchetti (1990).

1 Introduction

Since a class of discounted infinite horizon stochastic games was first introduced by Shapley (1953), and extended to more general n-player games

∗We thank Rabah Amir, Andy Atkeson, Frank Page, Adrian Peralta-Alva, Ed Prescott, and Manuel Santos for helpful discussions during the writing of this paper. This is a preliminary draft of the paper. Please do not recirculate, rather write the authors for the most recent draft. All the usual caveats apply.
†Institute of Mathematics, Wrocław University of Technology, 50-370 Wrocław, Poland.
‡Department of Economics, Arizona State University, USA.
§Department of Theoretical and Applied Economics, Warsaw School of Economics, Warsaw, Poland. Address: al. Niepodległości 162, 02-554 Warszawa, Poland. E-mail: lukasz.wozny@sgh.waw.pl.
subsequently (e.g., Fink (1964) or Sobel (1971)), the question of existence and characterization of stationary Markov Nash equilibrium (henceforth, SMNE) has been the subject of extensive study in game theory.\footnote{For example, see Raghavan, Ferguson, Parthasarathy, and Vrieze (1991) or Neyman and Sorin (2003) for an extensive survey of results, along with references.} Further, and perhaps more central to motivation of this paper, stochastic games have also become of fundamental tool for studying dynamic economic models in the presence of limited commitment among economic agents. Such limited commitment frictions arise naturally in diverse fields of economics, with economic applications including work in: (i) dynamic political economy (e.g., see Lagunoff (2009), and reference contained within), (ii) dynamic search with learning (e.g., see Curtat (1996) and Amir (2005)), (iii) equilibrium models of stochastic growth without commitment (e.g., Majumdar and Sundaram (1991) and Dutta and Sundaram (1992) and Amir (1996) for classic fish war problem and stochastic altruistic growth models), (iii) dynamic oligopoly models (see Rosenthal (1982), Cabral and Riordan (1994) or Pakes and Ericson (1998)), (iv) dynamic negotiations with status quo (see Duggan and Kalandrakis (2007)), (v) international lending and sovereign debt (Atkeson, 1991), (vi) optimal Ramsey taxation (Phelan and Stacchetti, 2001), (vii) models of savings and asset prices with hyperbolic discounting (Harris and Laibson, 2001), among others.\footnote{Also, for an excellent survey including other economic applications of stochastic games, please see Amir (2003)}

In economic applications of stochastic games, the issues of central concern have been broader than in the game theory literature, where the paramount question of focus has been on weakening conditions for the existence of subgame perfect and/or Markovian equilibrium. In particular, in economic applications of stochastic games, researchers are concerned also with the computational aspects of the economic model at hand, often requiring researchers to simulate of approximate equilibria to assess the quantitative importance of such dynamic/time consistency problems (e.g., for calibration exercises in applied macroeconomics), or to construct approximate equilibria to estimate the deep structural parameters of the game (as, for instance, in the recent work in empirical industrial organization.) For finite games\footnote{By "finite game" we mean a dynamic/stochastic game with a (a) finite horizon or (b) finite state and strategy space game. By an "infinite game", we mean a game where either (c) the horizon is countable (but not finite), or (d) the action/state spaces are a continuum. We shall focus on stochastic games where both (c) and (d) are present.}, the questions of existence and computation of SMNE has been substantially resolved.\footnote{For example, relative to existence, see Federgruen (1978) and Rieder (1979) (or more recently Chakrabarti (1999)); for computation of equilibrium, see Herings and Peeters (2004) concerning numerical algorithm for computation of equilibria; and, finally, for estimation of deep parameters, see Pakes and Pollard (1989); or more recently Aguirregabiria and Mira (2007), Pesendorfer and Schmidt-Dengler (2008), Pakes, Ostrovsky, and Berry (2007) and Doraszelski and Satterthwaite (2010).} Unfortunately, for infinite games, although the equilibrium ex-
istence question has received extensive attention, two additional questions of concern: (i) establishing results on the computation, approximation, and estimation of stochastic equilibrium and (ii) equilibrium comparative statics on the space of games, has yet to be satisfactorily resolved.

The aim of this paper is to address all of these concerns (i.e., existence, computation, and equilibrium comparatives statics) relative to SMNE for stochastic games within a single unified constructive approach. The unification of such results is important. For example, as approximation procedures that researchers typically employ are iterative, the need for characterizing accurate approximations of actual solutions to be “limits” of approximations of theoretical existence constructions has an obvious appeal (as they suggest direct methods for numerical implementation). Additionally, sensitivity analysis (relative to the set of SMNE) can be used not only to develop estimation/calibration procedures to assess the statistical significance of the economic model at hand, but also provide a method for understanding the role that variations in deep parameters play on the structure of economic equilibria.

In this paper, from a theoretical perspective, we study the existence question for a class of discounted, infinite horizon, stochastic games using monotone iteration methods. To obtain the needed monotonicity (of operators), as has been done in much of the existence literature, we focus on stochastic games with strategic complementarities. Unlike all existence work, though, our monotone iterative methods apply for the infinite horizon game. This allows us to obtain results for the infinite horizon stochastic game that have only been obtained in previous work for finite horizon stochastic games. Additionally, we can also produce conditions where infinite horizon SMNE are the limits of finite horizon stochastic games. This allows one to appeal to truncation/duality methods, in principle, as a method for constructing a stable set of SMNE in the deep parameters of the economy. Perhaps most importantly, we are able to give sufficient conditions under which our monotone iterative techniques are able to compute both equilibrium values and pure strategy Nash equilibrium from particular initial value/strategy pairs. As all of our constructions are based exclusively on successive approximation schemes (both for value function and strategies), aside from suggesting direct numerical implementations for computing approximation solutions for particular SMNE, they also allows us to provide monotone comparative statics on the space of primitive data for a class of infinite games.

Our methods are valid under very reasonable assumptions, and we manage to prove existence of a SMNE using different (be related) conditions than those used in previous work (e.g., Curtat (1996), Amir (2002, 2005) or Nowak (2007)). Under slightly stronger assumptions, we can then show that a set of Lipschitz continuous stationary Markov Nash equilibrium values form a complete lattice of continuous functions, that can be uniformly approximated as the limit of sequences generated by iterations on our fixed
point operator. Finally, we relate our methods (which are defined in function spaces endowed with pointwise partial orders) to the collection of recently proposed "correspondence-based" monotone continuation methods, which have extended the Kydland-Prescott/Abreu-Pearce-Stacchetti (henceforth KP/APS) approach where operators map in spaces of correspondences ordered under set inclusion. As we work exclusively in function spaces, where partial orders and order topologies are compatible (relative to convergence and continuity arguments), our methods avoid many of the difficult problems associated with numerical implementations of set-approximation (e.g., rigorously approximating measurable equilibrium selections), and the lack of characterization of the set of equilibrium pure strategies that support sustainable equilibrium values, both often being serious challenges for the KP/APS approach. Further, they avoid the use of nonconstructive topological arguments, where questions of approximation and equilibrium comparative statics are very difficult to answer.

A classical result on the existence for a subgame perfect equilibrium in infinite horizon stochastic games was first offered by Mertens and Parthasarathy (1987), and later modified, by other authors (e.g, Solan (1998) or Maitra and Sudderth (2007)). Motivated by the discussion in Maskin and Tirole (2001), and concentrating on the existence and properties of SMNE, authors have analyzed the structure of equilibria for the infinite horizon game by appealing to equilibrium in the so-called auxiliary (or, super) one period \( n \)-player game \( G_v^s \), where payoffs depends on a weighted average of (i) current within period payoffs, and (ii) a vector of expected continuation values \( v \), with weights given by a discount factor. Specifically, if \( s \in S \) is the state of the game, and if we denote by \( \Pi(v, s) \) a vector of values generated by any Nash equilibrium of \( G_v^s \), then it turns out, to show the existence of a stationary Markov equilibrium value in the original discount infinite horizon stochastic game, it suffices to show existence of a (measurable) "fixed point" selection in this auxiliary game (i.e., a measurable selection \( v^*(s) \in \Pi(v^*, s) \)). The existence of such fixed point in this problem has typically been achieved in the literature by developing applications of nonconstructive topological fixed point theorems on some nonempty, compact and (locally) convex space of value functions \( V \) (e.g., Schauder-Tychonoff’s generalization of Brouwer, or Fan-Glicksberg’s generalization of Kakutani fixed point theorems). To complete the proof of existence with this approach, one must prove the existence of a measurable Nash equilibrium of \( G_v^s \) that can be associated with any

\(^5\)For example, for the original ideas of the KP/APS approach, see the seminal work of Kydland and Prescott (1977, 1980), Abreu, Pearce, and Stacchetti (1990). See also the abstract approach defined in Mertens and Parthasarathy (1987). For more recent (and related) extensions of this approach, see Atkeson (1991), Pearce and Stacchetti (1997), Phelan and Stacchetti (2001) and Chakrabarti (2003), among others.

\(^6\)See also Harris, Reny, and Robson (1995) for a related argument on subgame perfect equilibria in repeated games with public randomization.
measurable fixed point \( v^*(s) \in \Pi(v^*, s) \) (i.e., you show any such measurable Nash equilibrium of the auxiliary game is a SMNE of a original infinite stochastic game).

For example, using this method, Nowak and Raghavan (1992) are able to show existence of a measurable value \( v^* \) of the original stochastic game using an application of Fan-Glicksberg’s generalization of Kakutani fixed point theorem. In particular, they deduce the existence of a correlated equilibrium of a stochastic game using a measurable selection theorem due to Himmelberg. This method has been further developed by Nowak (2003), where by adding a specific type of stochastic transition structure for the game (e.g., assuming a particular mixing structure), the existence of a measurable Markov stationary equilibrium is obtained. Here let us just mention that, one of our main assumptions in this paper, is to adopt a similar mixing restriction of stochastic transition as Nowak (2003).

When using the auxiliary game approach, it bears mentioning that one can make significant progress solving this problem if one can link a continuation value, say \( v' \), to a particular equilibrium value \( v(\cdot) \in \Pi(v', \cdot) \) from the same set of values (i.e., one can define a single-valued self map \( T_v(\cdot) = \Pi(v, \cdot) \) on a nonempty, convex and compact function space, where \( \Pi \) is some selection from \( \Pi \)). There are many approaches in the literature for using this idea. One way for obtaining such a selection structure is to assume that the auxiliary game has a unique equilibrium (and, hence, a unique equilibrium value \( \Pi(v, \cdot) \in CM \), where \( CM \) is the set of Lipschitz continuous functions on the state space \( S \)). This is precisely the approach that has been recently taken by many authors in the literature (e.g., Curtat (1996) and Amir (2002)). Conditions required to apply this argument are strong, requiring very restrictive assumptions on the game’s primitives involving strong concavity/diagonal dominance of the game, Lipschitzian structure for payoffs/transition structure in the auxiliary game, as well as stochastic supermodularity conditions. If these conditions are present, one can show the existence of monotone, Lipschitz continuous stationary equilibrium via a nonconstructive topological argument (e.g., Schauder’s theorem). A related, although more general, approach to this problem as recently been proposed by Nowak (2007). In Nowak’s approach, based on results of Nowak and Raghavan (1992) and Nowak (2003), he is able to prove an existence of a stationary Markov equilibrium where the key restrictive assumptions are placed on transition probability structure rather than preferences.

Another interesting idea for obtaining the needed structure to resolve these existence issues for the original game via the one-shot auxiliary game is to develop a procedure for selecting from from II an upper semi-continuous, increasing function on a compact interval of the real line and observing that a set of such functions is weak* compact).\(^7\) This approach was first explored

\(^7\)In our paper we use increasing / decreasing terms in their week meaning.
by Majumdar and Sundaram (1991) and Dutta and Sundaram (1992) in the important class of resource extraction games. In both of these papers, the existence of increasing, semi-continuous Markov stationary (symmetric) equilibria has been established. More recently, Amir (2005) has generalized this argument to a class of stochastic supermodular games, where both values and pure strategies for SMNE are shown to exist in spaces of increasing, upper semi-continuous functions. One common limitation of this purely topological approach is that to date, the authors have severely restrict the state space of the game (e.g., to being a compact interval of the real line), as well as requiring a great deal of complementarity between actions and states (i.e., assumptions consistent with the existence of monotone Markov equilibrium).

A final and somewhat related approach, to the existence and computation problem is the strategic dynamic programming approach, first proposed by Kydland and Prescott (1977, 1980) and Abreu, Pearce, and Stacchetti (1990) and since adapted to dynamic games in many papers (e.g., Atkeson (1991), Pearce and Stacchetti (1997), and Phelan and Stacchetti (2001), among others). Strategic dynamic programming is a correspondence-based dynamic programming approach, where dynamic incentive constraints are handled using correspondence-based arguments. In this approach, for each state $s \in S$ agents play a one-shot stage game with the continuation structure in the game parameterized by a correspondence of contiuation values, say $v' \in \mathcal{V}(S)$ where $\mathcal{V}(S)$ is the space of nonempty, bounded, upper-semicontinuous correspondences (for example). Imposing incentive constraints on deviations of the stage game with continuation $v'$, a natural monotone (under set inclusion) mapping $\mathcal{B}$ can be defined that transforms $\mathcal{V}(S)$ (and, hence, admits bounded, Borel measurable selections $v^*$ as selections). By iterating on a "value correspondence" operator from a "greatest" element of $\mathcal{V}(S)$ (that is mapped down under set inclusion) and appealing to standard "self-generation" arguments, it can be shown that a limiting of this decreasing collection of subsets consisting of all sustainable values in the game can be constructed, with convergence in the Hausdorff topology (and, equivalently in this context, the Scott topology), to a greatest fixed point of $\mathcal{B}$. This fixed point turns out to be the sustainable value correspondence in game, with a SMNE being any measurable selection from this limiting correspondence of values.

In this paper, we concentrate on an infinite horizon, stochastic games with (within period$^9$) complementarities and positive externalities (analyzed by Amir, Curtat or Nowak). Our methods obviously apply for finite horizon games; but for such games, our existence and comparative statics results

---

$^8$See also Amir (1996) for a related result for a stochastic common resource extraction game using transition structures that satisfy stochastic convexity conditions.

$^9$By this we mean supermodularity of a auxiliary game. See Echenique (2004) for a notion of a supermodular extensive form games.
can be obtained under weaker conditions. As will be clear in the sequel, our port of departure from some of the existing literature is that we adopt the specific noise structure introduced by Nowak (2003, 2006). Although this noise structure has been already used in existing literature extensively (e.g. in (Nowak, 2007)), our central focus is on characterizing appropriate monotone operators on the space of values (and, in some cases, pure strategies), where we manage to solve many difficulties obstructing the above mentioned authors relative to constructive applications of relevant fixed point theory. One critical observation that leads to our results is that under our assumptions, we are always able to select single-valued operators \( T \) that are monotone. So, in essence, the existence part of the argument is obtained with a fixed point theorem for monotone operators on a particular poset, while the computational procedures are implementations of the fixed point theorems of Amann (1976).

Additionally apart from our main results, in this paper, we construct a new KP/APS method that extends this existing strategic dynamic programming approach. A key difference between our approach and that in the existing literature is that our operator works in function spaces. Specifically, let \( \mathcal{V} \) be a function space of bounded measurable value functions, partially ordered under set inclusion. Using the auxiliary game, for a subset of values \( W \in \mathcal{V} \) in a function space of values \( \mathcal{V} \), we construct an KP/APS operator defined by \( B(W) = \{ w \in \mathcal{V} : w(\cdot) \in \Pi(v, \cdot), v \in W \in \mathcal{V} \} \), that transforms the \( \mathcal{V} \) into itself. As with the traditional KP/APS method, this operator is monotone under set inclusion. Therefore, the fact that \( B : \mathcal{V} \rightarrow \mathcal{V} \) has a greatest fixed point follows from Tarski’s theorem. Further, this greatest fixed point is the set of all sequential equilibria values \( \mathcal{V}^* \) in the function space \( \mathcal{V} \). Finally, appealing to "self-generation", and noting \( B(W) \) is order continuous in, successive approximations from an initial interval \( W_0 = \mathcal{V} \) generates a decreasing chain of subsets \( B^n(W_0) \) that converge (in this case, in order and Hausdorff) to \( W^* \), the greatest fixed point of \( B \). A central motivation for our construction is Mertens and Parthasarathy (1987) in their study of sub-game perfect equilibria (also, see Chakrabarti (1999) who extend of their results to SMNE). Although neither of these paper are able to achieve a constructive result under their very general conditions, given our slightly stronger assumptions (especially, our conditions of stochastic transitions), monotone iterative methods have limits in the set, and hence, they yield new constructive fixed point results.

Both of our approaches, selection and KP/APS, can be seen therefore as a link between selection methods used by Amir, Curtat, Nowak, a function-based implementation of the monotone operators methods used in papers appealing to the KP/APS approach.

The rest of the paper is organized as follows. Section 2 states the formal definition of an infinite horizon, stochastic game. Our main results (on existence and equilibrium approximation) can be found in section 3. In
section 4 we present related comparative statics and equilibrium dynamics results. For comparison in section 5 we show how KP/APS method can be used to show existence of a sequential Nash equilibrium. Finally in section 6 on particular examples, we show and discuss possible application of our results.

2 The Class of Stochastic Games

We consider a $n$-player, discounted infinite horizon stochastic game in discrete time. The primitives of the game are given by the tuple $\{S, (A_i, \tilde{A}_i, \beta_i, u_i)_{i=1}^n, Q, s_0\}$, where $S$ is a state space, $A_i \subset \mathbb{R}^{k_i}$ player $i$ action space with $A = \times_i A_i$, $\tilde{A}_i(s) \subset A_i$ its set of feasible actions in state $s$, $\beta_i$ is the discount factor for player $i$, $u_i : S \times A \to \mathbb{R}$ is the one-period payoff function, and $s_0 \in S$ the initial state. For each $s \in S$, the set of feasible actions $\tilde{A}_i(s)$ is a compact Euclidean interval in $\mathbb{R}^{k_i}$, where the state space $S$ is a compact interval $[0, \bar{S}] \subset \mathbb{R}^k$. By $Q$, we denote a transition function that specifies for a given current state $s \in S$ and a current action $a \in A$, a probability distribution over the next period states $s' \in S$.

A formal definition of a (Markov, stationary) strategy, payoff, and a Nash equilibrium in the infinite horizon stochastic, $n$-player game can be stated as follows. A strategy for a player $i$ is denoted by $\Gamma_i = (\gamma_i^1, \gamma_i^2, \ldots)$, where $\gamma_i^t$ specifies an action to be taken at stage $t$ as a function of history of all states $s^t$ and actions $a^t$ taken as of stage $t$. If a strategy depends on a partition of histories limited to the current state $s_t$, then the resulting strategy is Markov. If for all stages $t$, a Markov strategy has $\gamma_i^t = \gamma_i$ , then this stationary strategy $\Gamma_i$ for player $i$ is denote by $\gamma_i$, and $\gamma_i$ is called a Markov-stationary strategy. For a strategy profile $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_n)$, and initial state $s_0 \in S$, the expected payoff for player $i$ can be denoted by:

$$U_i(\Gamma, s_0) = (1 - \beta_i) \sum_{t=0}^{\infty} \beta_i^t \int u_i(s_t, a_t)dm_i^t(\Gamma, s_0),$$

where $m_i^t$ is the stage $t$ marginal on $A_i$ of the unique probability distribution (given by Ionescu–Tulcea’s theorem) induced on the space of all histories for $\Gamma$. A strategy profile $\Gamma^* = (\Gamma^*_i, \Gamma^*_{-i})$ is then a Nash equilibrium if $\Gamma^*$ is feasible, and for any $i$, and all feasible $\Gamma_i$, we have

$$U_i(\Gamma^*_i, \Gamma^*_{-i}, s_0) \geq U_i(\Gamma_i, \Gamma^*_{-i}, s_0).$$

3 Existence of SMNE via Monotone Operators

The aim of the paper is to not only prove the existence of a stationary Markov Nash Equilibrium, but to also provide successive approximation
schemes for constructing particular elements of this equilibrium set. The method we develop in this section of the paper is essentially a value iteration procedure restricted to operators that map functions to functions ((as opposed, for example, to strategic dynamic programming methods such as Kydland and Prescott (1980) or Pearce and Stacchetti (1997) that map correspondences to correspondences). We address however correspondence-based methods in section 5 of the paper.

To obtain our construction in the next two assumptions, we require some initial conditions on the primitives of the game. Our assumptions here are closely related to those made in recent work by Amir (2005) and Nowak (2007), but do have some important differences.

**Assumption 1 (Preferences)** For \( i = 1, \ldots, n \) let:

- \( u_i \) be continuous on \( A \) and measurable on \( S \), with \( u_i(s, a) \leq \pi \),
- \( (\forall a \in A) u_i(0, a) = 0 \),
- \( u_i \) be increasing in \( a_{-i} \),
- \( u_i \) be supermodular in \( a_i \) for each \( (a_{-i}, s) \), and has increasing differences in \( (a_i; a_{-i}) \),
- for all \( s \in S \) the sets \( \tilde{A}_i(s) \) are nonempty, compact intervals and \( s \rightarrow \tilde{A}_i(s) \) is a measurable correspondence.

**Assumption 2 (Transition)** Let \( Q \) be given by:

- \( Q(\cdot\mid s, a) = g_0(s, a)\delta_0(\cdot) + \sum_{j=1}^{L} g_j(s, a)\lambda_j(\cdot\mid s) \), where
- for \( j = 1, \ldots, L \) the function \( g_j : S \times A \rightarrow [0, 1] \) is continuous on \( A \) and measurable on \( S \), increasing in \( (s, a) \), supermodular in \( a \) for fixed \( s \), and \( g_j(0, a) = 0 \) (clearly \( \sum_{j=1}^{L} g_j(\cdot) + g_0(\cdot) \equiv 1 \)),
- \( (\forall s \in S, j = 1, \ldots, L) \lambda_j(\cdot\mid s) \) is a Borel transition probability on \( S \),
- \( (\forall j = 1, \ldots, L) \) function \( s \rightarrow \int_S v(s')\lambda_j(ds'\mid s) \) is measurable and bounded for any measurable and bounded \( v \),
- \( \delta_0 \) is a probability measure concentrated at point 0.

Our assumptions on preferences and stochastic transitions are weaker than those found in many important papers in the existing literature. Relative to Curtat and Amir (see (Curtat, 1996) or (Amir, 2002)), first notice that not only do we not require the payoffs or the transition probabilities
to be smooth (e.g., twice continuously differentiable on an open set containing \( S \times A \)). We also weaken the implied assumption on Lipschitz continuity of the functions \( u_i \) and \( g_j \) that is used in those papers. Additionally we do not impose conditions on payoffs and stochastic transitions that imply "doubly increasing differences" on each player's objective functions in actions and states as both of these authors do (for example, we do not assume any increasing differences between actions and states).

On the other hand, we do impose a very important condition that is stronger than these two papers; namely, a mixing condition on the transitions \( Q \). In particular, we assume the transition structure induced by \( Q \) can be represented as a convex combination of \( L + 1 \) measures, of which one measure is a delta Dirac concentrated at 0. As a result, with probability \( g_0 \), we set the next period state to zero; with probability \( g_j \), the distribution is drawn from the nondegenerate distribution \( \lambda_j \) (where, in this latter case, this distribution does not depend on the vector of actions \( a \), but is allowed to depend on the current state \( s \)). Also, although we assume each \( \lambda_j \) is stochastically ordered relative to the Dirac delta \( \delta_0 \), we do not impose stochastic orderings among the various measures \( \lambda_j \). This "mixing" assumption for transition probabilities has been appealed to extensively in the literature. For example, it was mentioned in Amir (1996), while studied systematically for broad classes of stochastic games first in Nowak (2003) and (almost everywhere) by Horst (2005). Further, in specific applications, it has recently been used to great advantage in various papers studying dynamic consistency problems in Balbus, Reffett, and Woźny (2009), and the references, therewithin). There are, however, important differences between functions \( g_j \) that we use in the current paper, and those used found in this existing literature. Specifically, as we are building our monotone continuation method in spaces of value functions, and not Euler equations (as, for example, in Balbus, Reffett, and Woźny (2009)), we do not require strict monotonicity of \( g_j \). This issue turns out to be important for generalities and results (e.g., see Balbus, Reffett, and Woźny (2009) where it is shown week monotonicity is needed to avoid trivial invariant distribution on bounded state spaces).

The paper by Amir (2005) is, perhaps, most closely related to our work. Relative to his paper, our requirements on payoffs and action sets are weaker (i.e., the feasible action correspondences \( \tilde{A}_i(s) \), and payoff/transition structures \( u_i \) and \( g_j \) are only required to be measurable with \( s \) (as opposed to upper semicontinuous as in Amir (2005))). Further, we also eliminate Amir’s assumption on increasing differences between players actions and the state variable (that is, for existence, we do not require monotone Markov equilibrium). Additionally, we do not require a single dimensional state space (as required for Amir’s existence argument for the infinite horizon version of his game). Finally, a critical difference between our work and Amir’s, however, is found in the specification of a transition \( Q \). In our work, we require a mixing specification for stochastic transition, while Amir for the infinite
horizon game requires a stochastic equicontinuity for the distribution function $Q$ relative to the actions $a$. These assumptions are incomparable, as one set of assumptions need not imply the other. For example, as we do not require any form of continuity of $\lambda_j$ with respect to the state $s$, we are not satisfying Amir’s assumption T1. On the other hand, Amir does not require any specific geometric form of a stochastic transition $Q$. Further, although we both require that the stochastic transition structure $Q$ is stochastically supermodular with $a$, we do not require increasing differences with $(a, s)$ as Amir does, nor stochastic monotonicity of $Q$ in $s$. Finally, let us mention that our assumptions are weaker than those studied in Nowak (2007).

To see the role of the particular assumptions on our results, the reader is referred to a proof of theorem 3.1 and the following discussion.

### 3.1 Existence and Computation of SMNE

We first prove existence of SMNE under our two assumptions, as well provide a successive approximation scheme for the pointwise approximation procedure for extremal SMNE. We then add structure to the primitives of the game, and provide uniform successive approximation methods for constructing stationary Markov equilibrium.

As in much of the work on stochastic supermodular games, our analysis of stationary Markov equilibria begins with equilibrium in an auxiliary static game. We first describe that auxiliary game in detail. For a vector of bounded, Borel-measurable continuation values $v = (v_1, v_2, \ldots, v_n) \in \mathcal{B}^n(S)$ with $v_i(0) = 0$ for each $i$, we analyze the auxiliary one-period, $n$-player game $G_{sv}$ with payoffs $\Pi_i$ and action set $A_i(s)$, namely:

$$\Pi_i(v_i, s, a_i, a_{-i}) := (1-\beta_i)u_i(s, a_i, a_{-i}) + \beta_i \sum_{j=1}^{L} g_j(s, a_i, a_{-i}) \int_S v_i(s') \lambda_j(ds'|s).$$

Equip the space $\mathcal{B}^n(S)$ with a pointwise partial product order.

We first prove three lemmata that are useful when constructing our monotone approach to the existence of SMNE in our game. The lemmata, in addition, also prove useful when characterizing iterative procedures for constructing least and greatest SMNE (relative to pointwise partial orders on $\mathcal{B}^n(S)$). The lemmata concern the structure of Nash equilibria (and their associated corresponding equilibrium payoffs values) in our auxiliary game $G_{sv}^\epsilon$. The first of three concerns the existence of monotone extremal Nash equilibrium in the auxiliary game.

**Lemma 3.1 (Monotone Nash equilibria in $G_{sv}^\epsilon$)** Under assumptions 1 and 2, for every $s \in S$ and value $v \in \mathcal{B}^n(S)$, the game $G_{sv}^\epsilon$ has the maximal Nash equilibrium $\overline{a}(v, s)$, and minimal Nash equilibrium $\underline{a}(v, s)$. Moreover, both equilibria are increasing in $v$. 


Proof: Without loss of generality fix \( s > 0 \). Define auxiliary one shot game, say \( \Delta(\tau) \), with an action space \( A \), and payoff function for player \( i \) given as

\[
H_i(a, \tau) := (1 - \beta_i)u_i(s, a_i, a_{-i}) + \beta_1 \sum_{j=1}^{L} \tau_{i,j}g_j(s, a_i, a_{-i}),
\]

where here \( \tau := [\tau_{i,j}]_{i=1,\ldots,n, j=1,\ldots,L} \in T := \mathbb{R}^{n \times L} \) is endowed with the natural pointwise order. Clearly, for each \( \tau \in T \), the game \( \Delta(\tau) \) is a supermodular, and satisfies all assumptions of Theorem 5 in Milgrom and Roberts (1990). Hence, there exists a complete lattice of Nash equilibrium, with greatest Nash equilibrium given by \( \overline{NE}\Delta(\tau) \), and least Nash equilibrium given by \( \underline{NE}\Delta(\tau) \). Moreover, for arbitrary \( i \), the payoff function \( H_i(a, \tau) \) has increasing differences in \( a_i \) and \( \tau \); hence, \( \Delta(\tau) \) also satisfies conditions of Theorem 6 in Milgrom and Roberts (1990). As a result, that equilibrium correspondence is Veinott strong set order ascending in \( \tau \), and both \( \overline{NE}\Delta(\tau) \) and \( \underline{NE}\Delta(\tau) \) are increasing in \( \tau \).

Step 2: For each \( s \in S \), the game \( G_s^v \) is a special case of \( \Delta(\tau) \) where \( \tau_{i,j} = \int_{S} v_i(s')\lambda_j(ds'|s) \). Therefore, by the previous step, its set of Nash equilibrium a complete lattice for each \( (v, s) \), and is Veinott strong set order ascending in \( v \), each \( s \in S \); hence, its least and greatest Nash equilibrium \( g(v, s) \) and \( \pi(v, s) \) are increasing in \( v \), for each \( s \in S \).

In the next lemma, for each extremal Nash equilibrium in state \( s \) with continuation value \( v \), we associate an equilibrium payoff. To do this, we begin by initially defining the following pair of values in a greatest (resp., least) best response to continuation values \( v \), in state \( s \):

\[
\Pi_i^*(v, s) := \Pi_i(v_i, s, \pi_i(v, s), \pi_{-i}(v, s))
\]

and similarly

\[
\Pi_i^*(v, s) := \Pi_i(v_i, s, a_i(v, s), a_{-i}(v, s)).
\]

We now have the following lemma:

**Lemma 3.2 (Monotone values in \( G_s^v \))** Under assumptions 1 and 2 we have: \( \Pi_i(v, s) \) and \( \Pi_i^*(v, s) \) are monotone in \( v \).

**Proof:** \( \Pi_i \) is increasing with \( a_{-i} \) and \( v_i \). For \( v_2 \geq v_1 \) by Lemma 3.1, we have \( a(v_2, s) \geq a(v_1, s) \). Hence,

\[
\Pi_i^*(v_2, s) = \max_{a_i \in A_i(s)} \Pi_i(v_2^i, s, a_i, a_{-i}(v_2^i, s)) \geq \max_{a_i \in A_i(s)} \Pi_i(v_1^i, s, a_i, a_{-i}(v_2^i, s)) \geq \max_{a_i \in A_i(s)} \Pi_i(v_1^i, s, a_i, a_{-i}(v_1^i, s)) = \Pi_i^*(v_1, s).
\]

(1)
A similar argument proves the monotonicity of $\Pi_i^*(v, s)$. ■

Finally, we construct a measurable equilibrium (and its associated equilibrium values) in the auxiliary game as a function of the state $s \in S$. To do this, define the vector of values $\Pi^*(v, s) = (\Pi_1^*(v, s), \Pi_2^*(v, s), \ldots, \Pi_n^*(v, s))$. Using these mappings, we can define a pair of extremal value operators that transform the space of bounded measurable values $\mathcal{B}^n(S)$; namely, a pair of operators $T(v)(s) = \Pi^*(v, s)$ and $T(v)(s) = \Pi^*(v, s)$, which are the "upper" and "lower" values in the auxiliary game. To show that $T$ and $T$ are well-defined transformations of $\mathcal{B}^n(S)$ we use the techniques introduced by Nowak and Raghavan (1992).

**Lemma 3.3 (Measurable equilibria and values of $G_v^s$)** Under assumptions 1 and 2 we have:

- $T : \mathcal{B}^n(S) \to \mathcal{B}^n(S)$ and $T : \mathcal{B}^n(S) \to \mathcal{B}^n(S)$,
- functions $s \to \pi(v, s)$ and $s \to a(v, s)$ are measurable for any $v \in \mathcal{B}^n(S)$.

**Proof:** For $v \in \mathcal{B}^n(S)$ and $s \in S$, define the function $F_v : A \times S \to \mathbb{R}$ as follows:

$$F_v(a, s) = \sum_{i=1}^{n} \Pi_i(v_i, s, a) - \max_{z_i \in A_i(s)} \Pi_i(v_i, s, z_i, a_{-i})$$

Observe that $F_v(a, s) \leq 0$. Now, consider the problem

$$\max_{a \in \times_i A_i(s)} F_v(a, s)$$

By assumption, the objective $F_v$ is a Carathéodory function, and (joint) feasible correspondence in $A(s) = \times_i \tilde{A}_i(s)$ is weakly-measurable. Therefore, by the measurable maximum theorem (e.g. theorem 17.18 in Aliprantis and Border (1999)), the argmax $N_v : S \to \times_i A_i(s)$ is measurable with nonempty compact values. Further, observe that $N_v(s)$, by definition, is a set of all Nash equilibria for the game $G_v^s$. Therefore, to finish the proof of our first assertion, for some player $i$, consider a problem $\max_{a \in N_v(s)} \Pi_i(v, s, a)$. Again, appealing to the measurable maximum theorem, the value function $\Pi_i^*(v, s)$ is measurable. A similar argument shows each $\Pi_i^*(v, s)$ is measurable. Therefore, we have for value operators $T : \mathcal{B}^n(S) \to \mathcal{B}^n(S)$ and $T : \mathcal{B}^n(S) \to \mathcal{B}^n(S)$.

To show a second assertion, for some player $i$, again consider a problem of $\max_{a \in N_v(s)} \Pi_i(v, s, a)$. Once again, appealing to the measurable maximum theorem, the selection $\pi(v, s)$ (respectively, $a(v, s)$) is measurable with $s$. ■
We are now prepared to prove the first main theorem of the paper on the existence of SMNE. We verify existence constructively by providing a successive approximation method that provides procedures for constructing SMNE pure strategies and values. To do this, letting $T^t(v)$ denote $t$-iteration/orbit of the operator $T(v)$ from $v$, we can iteratively define a sequence of lower (resp., upper) bounds for equilibrium values $\{v^t\}_{t=0}^\infty$ (resp., $\{w^t\}_{t=0}^\infty$) where $v^{t+1} = T(v^t)$ for $t \geq 1$ from initial guess $v^1(s) = (0, 0, ..., 0)$ (resp., where $w^{t+1} = T(w^t)$ from initial guess $w^1(s) = \pi$). Further, for both lower (resp., upper) value iterations, we can also define sequences of associated pure Nash equilibrium strategies $\{\psi^t\}_{t=0}^\infty$ (resp., $\{\psi^t\}_{t=0}^\infty$) associated with the sequence $\{v^t\}_{t=0}^\infty$ (resp., $\{w^t\}_{t=0}^\infty$) that are each related by the iterations $\phi^t = \varphi(s, v^t)$ (resp., $\psi^t = \pi(s, w^t)$).

**Theorem 3.1 (The Successive approximation of SMNE)** Under assumptions 1 and 2 we have

1. for fixed $s \phi^t(s)$ and $\psi^t(s)$ are increasing sequences and $\psi^t(s)$ and $w^t(s)$ are decreasing sequences,

2. for all $t$ we have $\phi^t \leq \psi^t$ and $v^t \leq w^t$,

3. the following limits exists: $(\forall s \in S) \lim_{t \to \infty} \phi^t(s) = \phi^*(s)$ and $(\forall s \in S) \lim_{t \to \infty} \psi^t(s) = \psi^*(s)$,

4. the following limits exists: $(\forall s \in S) \lim_{t \to \infty} v^t(s) = v^*(s)$ and $(\forall s \in S) \lim_{t \to \infty} w^t(s) = w^*(s)$

5. $\phi^*$ and $\psi^*$ are stationary Markov Nash equilibria in the infinite horizon stochastic game.

**Proof:** Clearly $\phi^1 \leq \phi^2$ and $v^1 \leq v^2$. Suppose $\phi^t \leq \phi^{t+1}$ and $v^t \leq v^{t+1}$. By the definition of the sequence $\{v^t\}$ and lemma 3.2, we have $v^{t+1} \leq v^{t+2}$. Then, by Lemma 3.1, definition of $\{\phi^t\}$, and the induction hypotheses, we obtain $\phi^{t+1}(s) = \varphi(v^{t+1}, s) \leq \varphi(v^{t+2}, s) = \phi^{t+2}(s)$. Similarly, we obtain monotonicity of $\psi^t$ and $w^t$.

---

10One key aspect of our work relative to Kydland-Prescott/Abreu-Pearce-Stacchetti (KP/APS) strategic dynamic programming arguments is that our methods constructive selections for both values and strategies via constructive procedures. When using KP/APS strategic dynamic programming methods, although set approximation methods even in the simplest stochastic games (e.g., cases of games where measurability is not an issue), constructive methods are only available for constructing the set of sustainable values in any state (not the strategies which support them, which can be pure or mixed strategies). In games with measurability issues, even the set of sustainable values are only measurable selections from a larger (measurable) correspondence; hence, not informative about how to construct equilibrium selections.
Proof of 2: Clearly, the thesis is satisfied for $t = 1$. By induction, suppose that the thesis is satisfied for some $t$. Since $v^t \leq w^t$, by Lemma 3.2, we obtain

$$v^{t+1}(s) = \Pi^*(v^t, s) \leq \Pi^*(w^t, s) \leq \Pi^*(w^t, s) = w^{t+1}(s).$$

Then, by Lemma 3.1, we obtain

$$\phi^{t+1}(s) = a(v^{t+1}, s) \leq a(w^{t+1}, s) \quad \text{and hence} \quad \leq a(w^{t+1}, s) = \psi^{t+2}(s).$$

Proof of 3-4: It is clear since for each $s \in S$, the sequences of values $v^t$, $w^t$ and associated pure strategies $\phi^t$ and $\psi^t$ are bounded. Further, by previous step, they are monotone.

Proof of 5: By definition of $v^t$ and $\phi^t$, we obtain

$$v_i^{t+1}(s) = (1 - \beta_i)u_i(s, \phi^t(s)) + \beta_i \sum_{j=1}^L g_j(s, \phi^t(s)) \int_S v_i^t(s')\lambda_j(ds'|s)$$

$$\geq (1 - \beta_i)u_i(s, a_i, \phi^t_{-i}(s)) + \beta_i \sum_{j=1}^L g_j(s, a_i, \phi^t_{-i}(s)) \int_S v_i^t(s')\lambda_j(ds'|s),$$

for arbitrary $a_i \in \tilde{A}_i(s)$. By continuity of $u_i$ and $g$ and by Lebesgue Domi-
nance Theorem, if we take a limit $t \to \infty$, we obtain

$$v_i^*(s) = (1 - \beta_i)u_i(s, \phi^*(s)) + \beta_i \sum_{j=1}^L g_j(s, \phi^*(s)) \int_S v_i^*(s')\lambda_j(ds'|s)$$

$$\geq (1 - \beta_i)u_i(s, a_i, \phi^*_{-i}(s)) + \beta_i \sum_{j=1}^L g_j(s, a_i, \phi^*_{-i}(s)) \int_S v_i^*(s')\lambda_j(ds'|s),$$

which, by lemma 3.3, means that $\phi^*$ is a pure stationary (measurable) Nash equilibrium, and $v^*$ is its associated (measurable) equilibrium payoff. Analogously, we have $\psi^*$ a pure strategy (measurable) Nash equilibrium, and $w^*$ its associated (measurable) equilibrium payoff.

We are also able to give pointwise bounds for successive approximations of any SMNE using upper and lower iterations built from the construction in Lemma 3.2 and Theorem 3.1. The result is stated as follows:
Theorem 3.2 (Pointwise Bounds for SMNE) Let assumptions 1 and 2 be satisfied and $\gamma^*$ be an arbitrary stationary Markov Nash equilibrium. Then, $(\forall s \in S)$, we have the pointwise bounds $\phi^*(s) \leq \gamma^*(s) \leq \psi^*(s)$. Further, if $\omega^*$ is equilibrium payoff associated with any stationary Markov Nash equilibrium $\gamma^*$, then $(\forall s \in S)$ we have the pointwise bounds $\nu^*(s) \leq \omega^*(s) \leq \psi^*(s)$.

Proof: Step 1: We prove the desired inequality for equilibria payoffs. Since $0 \leq \omega^* \leq \bar{u}(s)$, hence by Lemma 3.2 and definition of $v^t$ and $w^t$ we obtain

$$v_1 \leq \omega^* \leq w_1.$$  

Suppose by induction that $v_t \leq \omega^* \leq w_t$. Again from Lemma 3.2 we have

$$v_{t+1} = \Pi^*(v_t, s) \leq \Pi^*(\omega^*, s) \leq \omega^*(s) \leq \Pi^*(\omega^*, s) \leq \Pi^*(w_t, s) = w_{t+1}.$$  

Taking a limit with $t$ we obtain desired inequality for equilibria payoffs.

Step 2: We show that similar inequality is satisfied for strategies. By previous step and Lemma 3.1 we obtain

$$\phi^*(s) = \underline{a}(v^*, s) \leq \underline{a}(\omega^*, s) \leq \gamma^*(s) \leq \bar{a}(\omega^*, s) \leq \bar{a}(w^*, s) = \psi^*.$$  

Before we continue, it is worth making a few remarks on how Theorems 3.1 and 3.2 relate to the known results in the existing literature. First, the existence result in Theorem 3.1 is obtained under a different assumptions than Curtat (1996) or Amir (2002, 2005). For example, we do require increasing differences between player’s actions and the state of the game. To accomplish this weakening of conditions, relative to Curtat (1996) or Amir (2002), we exploit the mixing conditions on the stochastic transition $Q$ in Assumption 2 that is assumed in neither of these two papers. The assumption though allows us to able a tractable version of the auxiliary game (relative to the infinite horizon stochastic game), which leads to obtain monotone iterative procedures for constructing equilibrium value functions and pure strategy equilibrium. Second, our existence result does not require Lipschitz continuity of payoffs and transitions (as functions of states and actions); hence, for example, this allows for period payoffs that are consistent with Inada type assumptions (which, for example, are also ruled out in the work of Horst (2005). Finally, we do not require monotonicity of a transition probability nor diagonal dominance type assumptions on both payoff and transition.
As compared to the results of Amir (2005), our results are stronger. In particular, (as we have strengthen the conditions on the noise, although we have weakened the monotonicity and complementarity conditions) aside from verifying the existence of SMNE, and providing methods for constructing them, we will be able to obtain monotone comparative statics results on the space of games in a moment for the infinite horizon game (which is not possible using Amir’s purely topological approach). Also, our state space allows to multidimensional decisions.

To better understand the differences in our approach with Amir’s, first observe Amir’s existence proof is based on a Schauder fixed point theorem. Specifically, Amir shows (weak*) continuity of a best response operator an a compact and convex set of monotone, upper semicontinuous strategies defined on the real line. For this result, he requires stochastic equicontinuity of a distribution \( Q \). On the other hand, we just construct a sequence of functions whose limit is a (fixed-point) value leading to SMNE. Therefore, as we do not use Schauder fixed point theorem, we do not need to require continuity of a best-response operator nor compactness of a particular strategy space. In our approach, the critical result is obtained in theorem 3.1, 3.2 (and later in theorem 3.3 or 3.4) concerning the monotonicity of our value function operator \( T \) in continuation values, for each state \( s \in S \).

Finally, as our limiting arguments are based on pointwise continuity (and, hence, order continuity, see Aliprantis and Border (1999) lemma 7.16), this allows use to equivalently state all our constructive results using fixed point theorems for \( \sigma \)- complete lattices (e.g., see Dugundji and Granas (1982), Theorem 4.1-4.2) where continuity is characterized in order topologies (e.g., a Scott topology with a countable basis). It is precisely here where our mixing assumption on the noise has its bite, as the assumption leads to a tractable form of monotonicity that is preserved to the infinite horizon game under assumption 2. For example, in Curtat (1996), one can only manage to show monotonicity of an operator \( T \) in the gradients \( \partial v \), where \( \partial \) denote a vector of (almost everywhere defined) differentials of \( v \).

We finish this section with an important remark for applications we discuss in section 6. Namely, as proofs of theorems 3.1 and 3.2 are conducted pointwise we can weaken restrictions placed on \( S \).

**Remark 1** Theorems 3.1 and 3.2 (and later theorems 4.1 and 4.2) works for compact posets \( S \) with the minimal element 0, not necessarily boxes in \( \mathbb{R}^k \).

### 3.2 Uniform Error Bounds for Lipschitz continuous SMNE

We now give two motivations for the next results. Firstly: we need some stronger assumptions to address the question of Lipschitz continuity of equilibrium strategies. Secondly: observe that the limits and approximation re-
sults provided by theorems 3.1 and 3.2 are only pointwise. Now we present
an important, from the computationally point of view, result on uniform
bound for a set of stationary Markov Nash equilibria. For this reason we

**Assumption 3** For all $i, j$:

- $u_i, g_j$ are twice continuously differentiable on an open set containing $S \times A$,
- $u_i$ satisfy a strong dominant diagonal condition in $a$ for fixed $s \in S, s > 0$,
- $g_j$ satisfy a dominant diagonal condition in $a$ for fixed $s \in S, s > 0$,
- for each $f$, the function $\eta^f_j(s) := \int_S f(s') \lambda_j(ds'|s)$ is Lipshitz-continuous\(^{11}\),
- $\bar{A}_i(0) = 0$.

Define a set $CM$ of $n$-tuples, increasing Lipschitz continuous (with some constant $K$) functions on $S$. Observe that $CM$ is a complete lattice when endowed with a partial order (and also convex and compact in the sup norm).

**Theorem 3.3 (Lipschitz continuity)** Let assumptions 1, 2, 3 be satisfied. Stationary Markov Nash equilibria $\phi^*, \psi^*$ and corresponding values $v^*, w^*$ are all Lipschitz continuous.

**Proof:** Using Gabay and Moulin (1980) result Curtat (1996) and Amir (2002) shows that under above assumptions the auxiliary game $G^v$ has a unique Nash equilibrium, i.e.: $a^*(v, s) := g(v, s) = \pi(v, s)$ and hence $\Pi^*(v, s) := \Pi^*(v, s) = \Pi^*(v, s)$. Moreover by Granot and Veinott (1985) results on doubly increasing differences $a^*(v, s)$ is Lipschitz continuous in $s$, with a constant $K$ independent on $v \in CM$.

By Lipschitz continuity of $u_i$ and $g_j$ and $\eta^v_i(s)$ we obtain Lipschitz continuity of $\Pi^* = \Pi^* = \Pi^*$ in $s$ uniformly of $v$. The maximal and minimal fixed point of increasing continuous operator $T(v)(s) = \Pi^*(v, s)$ defined on $CM$ are $w^*$ and $v^*$ respectively and belong to $CM$. The corresponding stationary Nash equilibria $\psi^*$ and $\phi^*$ are hence also Lipschitz continuous.

We can now provide conditions under which we can provide uniform approximation results for SMNE. The results uses a version of Amann’s

\(^{11}\)This condition is satisfied if each of measures $\lambda_i(ds'|s)$ has a density $\rho_i(s'|s)$ and the function $s \to \rho_i(s'|s)$ is Lipshitz continuous uniformly in $s'$. 

18
theorem (e.g., Amann (1976), Theorem 6.1) to characterize least and greatest SMNE via successive approximations. Further, as a corollary of Thorem 3.1, we also obtain their associated value functions. For this argument, define \( \mathbf{0} \) to be an \( n \)-tuple of function identically equal to 0 for all \( s \in S \). The result is then stated as follows:

**Corollary 3.1 (Uniform approximation of extremal SMNE)** Let assumptions 1, 2 and 3 be satisfied. Then \( \lim_{t \to \infty} ||T^t \mathbf{0} - v^*|| = 0 \) and \( \lim_{t \to \infty} ||T^t \pi - w^*|| = 0 \), with \( \lim_{t \to \infty} ||\phi^t - \phi^*|| = 0 \) and \( \lim_{t \to \infty} ||\psi^t - \psi^*|| = 0 \).

Notice, the above corollary assures that the convergence in theorem 3.1 is uniform. We also obtain a stronger characterization the set of SMNE in this case. In particular, we show the set of SMNE equilibrium value functions is a complete lattice.

**Theorem 3.4 (Complete lattice structure of SMNE values set)** Under assumptions 1, 2 and 3 the set of stationary Markov Nash equilibrium values \( v^* \) in \( CM \) is a nonempty complete lattice.

**Proof:** On \( CM \) define a function \( T(v)(s) = \Pi^*(v, s) \)and follow Curtat (1996) to show that \( T : CM \to CM \) is continuous. Moreover \( T \) is increasing on \( CM \) and hence by Tarski (1955) theorem has a nonempty complete lattice of fixed points \( FP(T) \). Each fixed point of \( T \) say \( v^*(\cdot) \in FP(T) \) corresponds to a unique stationary Markov Nash equilibrium \( a^*(v^*, \cdot) \).

The above result provides a further characterization of a SMNE strategies as well as their corresponding set of equilibrium value functions. From a computational point of view, not only are the extremal values and strategies Lipschitzian (as known from previous work), they can also be uniformly approximated by iterations on our fixed point operator. Also observe that the set of SMNE in \( CM \) is not necessarily a complete lattice.

### 4 Monotone Comparative Dynamics

In their papers Curtat (1996) and Amir (2002) provide further characterizations of the stationary Markovian Nash equilibrium strategies. Importantly, although these papers make very strong assumptions concerning the stochastic supermodular structure of the game, they do not address, the question of monotone comparative dynamics, nor the question of structure of stationary Markov equilibrium or ordered stochastic invariant distributions. We address these questions in this section of the paper.

To do this, we start with a useful lemma:
Lemma 4.1 Let $X$ be a lattice, $Y$ be a poset. Assume (i) $F : X \times Y \to \mathbb{R}$ and $G : X \times Y \to \mathbb{R}$ have increasing differences, (ii) that $\forall y \in Y$, $G(\cdot, y)$ and $\gamma : Y \to \mathbb{R}$ are increasing functions. Then, function $H$ defined by $H(x, y) = F(x, y) + \gamma(y)G(x, y)$ has increasing differences.

Proof: Under the hypotheses of the lemma, it suffices to show that $\gamma(y)G(x, y)$ has increasing differences (as increasing differences is a cardinal property and closed under addition). Let $y_1 > y_2$, $x_1 > x_2$ and $(x_i, y_i) \in X \times Y$. By that hypothesis of increasing differences of $G$, and monotonicity of $\gamma$ and $G(\cdot, y)$, we have

$$\gamma(y_1)(G(x_1, y_1) - G(x_2, y_1)) \geq \gamma(y_2)(G(x_1, y_2) - G(x_2, y_2)).$$

Therefore,

$$\gamma(y_1)G(x_1, y_1) + \gamma(y_2)G(x_2, y_2) \geq \gamma(y_1)G(x_2, y_1) + \gamma(y_2)G(x_1, y_2).$$

Given this lemma, we can now provide conditions under which our class of games exhibits equilibrium monotone comparative statics of extremal fixed point values $v^*, w^*$, as well as the corresponding extremal equilibria $\phi^*, \psi^*$. We can also address the question of ordered equilibrium dynamics. With this question in mind, consider a parametrization of our stochastic game by $\theta \in \Theta$, where $\Theta$ is a partially ordered set. We can view $\theta$ as representing the deep parameters of the space of games from which the period preferences $u_i$, the stochastic transitions $g_j$ and $\lambda_j$, feasibility set $A_i$ are chosen. Alternatively, we can think of elements $\theta$ as being policy parameters of the environment governing the setting of taxes or subsidies for a dynamic policy game. Along those lines, we provide parameterized versions of Assumptions 1 and 2 as follows:

Assumption 4 (Parameterized preferences) For $i = 1, \ldots, n$ let:

- $u_i : S \times A \times \Theta \to \mathbb{R}$ be measurable and $u_i(\cdot, s, \theta)$ continuous on $A$ for any $s \in S, \theta \in \Theta$ with $u_i(\cdot) \leq \pi$,
- $(\forall a \in A, \theta \in \Theta) u_i(0, a, \theta) = 0$,
- $u_i$ be increasing in $(s, a_{-i}, \theta)$,
- $u_i$ be supermodular in $a_i$ for fixed $(a_{-i}, s, \theta)$, and has increasing differences in $(a_i; a_{-i}, s, \theta)$,
• for all \( s \in S, \theta \in \Theta \), the sets \( \tilde{A}_i(s, \theta) \) are nonempty, measurable, compact intervals and a measurable multifunction that is both ascending in Veinott’s strong set order\(^{12}\), and expanding under set inclusion\(^{13}\) with \( \tilde{A}_i(0, \theta) = 0 \).

**Assumption 5 (Parameterized transition)** Let \( Q \) be given by:

- \( Q(s, a, \theta) = g_0(s, a, \theta)\delta_0(\cdot) + \sum_{j=1}^{L} g_j(s, a, \theta)\lambda_j(\cdot|s, \theta), \) where
- for \( j = 1, \ldots, L \) the measurable function \( g_j : S \times A \times \Theta \to [0, 1] \) is continuous with \( a \) for a given \( s, \theta \), increasing in \( (s, a, \theta) \), supermodular in \( a \) for fixed \( (s, \theta) \), and has increasing differences in \((a; s, \theta)\) and \( g_j(0, a, \theta) = 0 \) (clearly \( \sum_{j=1}^{L} g_j(\cdot) + g_0(\cdot) \equiv 1 \)),
- \((\forall s \in S, \theta \in \Theta, j = 1, \ldots, L) \lambda_j(\cdot|s, \theta) \) is a Borel transition probability on \( S \), with each \( \lambda_j(\cdot|s, \theta) \) stochastically increasing with \( \theta \) and \( s \),
- \((\forall j = 1, \ldots, L) \) function \( (s, \theta) \to \int_S v(s')\lambda_j(ds'|s, \theta) \) is measurable for any measurable and bounded \( v \),
- \( \delta_0 \) is a probability measure concentrated at point 0.

Notice, in both of these assumptions, we have added now increasing difference assumptions between actions and states (as, for example, in Curtat (1996) and Amir (2002)). We can now prove a result on monotone equilibrium comparative statics for the set of SMNE. To do this, we first introduce some notation. For a stochastic game fixed at parameter \( \theta \in \Theta \), denote the least and greatest equilibrium values, respectively, as \( v^*_\theta \) and \( w^*_\theta \). Further, we each of these extremal values, denote the associated least and greatest SMNE pure strategies, respectively, as \( \phi^*_\theta \) and \( \psi^*_\theta \). Our first monotone comparative statics theorem is given in the next theorem:

**Theorem 4.1 (Monotone comparative statics)** Let assumptions 4 and 5 be satisfied. Then, the extremal equilibrium values \( v^*_\theta(s) \), \( w^*_\theta(s) \) are increasing on \( S \times \Theta \). In addition, the associated extremal pure strategy stationary Markov Nash equilibrium \( \phi^*_\theta(s) \) and \( \psi^*_\theta(s) \) are increasing on \( S \times \Theta \).

**Proof:** Step 1. Let \( v_\theta \) be a such function that \((s, \theta) \to v_\theta(s) \) be increasing.

By assumption 5 and lemma 4.1, the payoff function \( \Pi_i(v_\theta, s, a, \theta) \) has increasing differences in \((a_i, \theta)\). Further, \( \Pi_i \) also clearly increasing differences

\(^{12}\)That is, \( \tilde{A}_i(s, \theta) \) is ascending in Veinott’s strong set order if for any \((s, \theta) \leq (s', \theta'), a_i \in \tilde{A}_i(s, \theta) \) and \( a'_i \in \tilde{A}_i(s', \theta') \) \( \Rightarrow a_i \land a'_i \in \tilde{A}_i(s, \theta) \) and \( a_i \lor a'_i \in \tilde{A}_i(s', \theta') \).

\(^{13}\)That is, \( \tilde{A}_i(s, \theta) \) is expanding if \( s_1 \leq s_2 \) and \( \theta_1 \leq \theta_2 \) then \( \tilde{A}_i(s_1, \theta_1) \subseteq \tilde{A}_i(s_2, \theta_2) \).
in \((a_i, a_{-i})\). As \(\tilde{A}(\cdot)\) is also ascending in Veinott’s strong set order, by Theorem 6 in Milgrom and Roberts (1990), the greatest and the smallest Nash equilibrium in the supermodular game \(G^*_{e, \theta}\) are increasing selections. By the same argument as in Lemma 3.2, under assumption that \(\tilde{A}(\cdot)\) are ascending under set inclusion, we obtain monotonicity of corresponding equilibria payoff.

Step 2. Note, for each \(\theta\), the stochastic parameterized game satisfies conditions of Theorem 3.1. Further, the initial values of the sequence of \(w^0_\theta(s)\) and \(v^0_\theta(s)\) (constructed in Theorem 3.1) does not depend on \(\theta\) neither \(s\); hence, by the previous step, each iteration of both sequences of values is increasing with respect to \((s, \theta)\). Further, each of the iterations of \(\phi^0_\theta(s)\) and \(\psi^0_\theta(s)\) are also increasing in \((s, \theta)\). Therefore, as the pointwise partial order is closed, the limits of these sequences preserve this partial ordering, and hence, the limits increase with respect to \((s, \theta)\).

In the literature on stochastic games with strategic complementarities, for infinite horizon games, we are not aware of any analog result concerning monotone equilibrium comparative statics as in Theorem 4.1. In particular, because of the non-constructive approach to the equilibrium existence problem (that is typically taken in the literature (e.g., Curtat (1996), Amir (2002), Amir (2005), and Nowak (2007))), for the infinite horizon game, it is difficult to obtain such a monotone comparative statics without fixed point uniqueness (e.g. Villas-Boas (1997) for the details). Therefore, a key innovation of our approach of the previous section is that for the special case that SMNE are monotone Markov processes, we are able construct a sequence of parameterized monotone operators whose fixed points are extremal equilibrium selections. As the method is constructive, this also allows use to compute directly the relevant monotone selections from the set of SMNE.

Finally, we state results on dynamics and invariant distribution started from \(s_0\) and governed by a SMNE and transition \(Q\). Before that let us mention that by our assumptions delta Dirac concentrated at 0 is an absorbing state and hence an invariant distribution. As a result we do not aim to prove existence of an invariant distribution but rather characterize a set of all invariant distributions and discuss conditions when it is not a singleton. For this reason let \(\theta\) be given and by \(s^f_t\) denote a process induced by \(Q\) and equilibrium strategy \(f\) i.e. \(s_0 = s^f_0\) is an initial value and for \(t > 0\) \(s^f_{t+1}\) has a conditional distribution \(Q(\cdot|s_t, f(s_t))\). For each equilibrium strategy \(f\) define an operator

\[
T^\theta_f(\eta)(A) = \int Q(A|s, f(s))\eta(ds).
\]

\(\eta^*\) is said to be invariant with respect to \(f\) if and only if it is a fixed point of
Finally by ≥ denote a partial order on a space of probability measures defined by first order stochastic dominance.

**Theorem 4.2 (Invariant distribution)** Let assumptions 4, 5 be satisfied with \( s \to \lambda_j(\cdot|s) \) stochastically continuous\(^{14}\) for any \( j \).

- Then sets of invariant distributions for processes \( s_t^\phi \) and \( s_t^\psi \) are chain complete posets (with both greatest and least elements) with respect to (first-) stochastic order.

- Let \( \eta(\phi^*) \) be the greatest invariant distribution with respect to \( \phi^* \) and \( \eta(\psi^*) \) the greatest invariant distribution with respect to \( \psi^* \). If the initial state of \( s_t^\phi \) or \( s_t^\psi \) is a Dirac delta in \( \bar{S} \) then \( s_t^\phi \) converge weakly to \( \eta(\phi^*) \) and \( s_t^\psi \) to \( \eta(\psi^*) \) respectively.

**Proof:** By theorem 4.1 both \( \phi^* \) and \( \psi^* \) are increasing functions. By Assumption 2 \( T_\phi^\phi(\eta)([s, \bar{S}]) = 1 \) for \( s \leq 0 \) and for \( s > 0 \):

\[
T_\phi^\phi(\eta)([s, \bar{S}]) = \sum_{j=1}^{L} g_j(s, \phi^*(s)) \int_S \lambda_j([s, \bar{S}]|s') \eta(ds').
\]

Since by assumption for each \( s \in S \) the function under integral is increasing, hence the whole statement is increasing whenever \( \eta \) is stochastically increasing. Moreover the family of probability measures on a compact state space \( S \) ordered by ≥ (first order stochastic dominance) is chain complete (as it is a compact ordered topological space, see Amann (1977), lemma 3.1 or corollary 3.2). Hence \( T_\phi^\phi \) satisfies conditions of Markowsky (1976) theorem (Theorem 9, Section 7). We conclude that the set of invariant distributions is a chain complete poset and further, greatest and least invariant distributions exists (see Amann (1977) theorem 3.3). The same way we show that thesis for operator \( T_\psi^\psi \).

To show the second assertion we prove that \( T_\phi^\phi(\cdot)(A) \) is weakly continuous (i.e. if \( \eta_n \to \eta \) weakly then \( T_\phi^\phi(\eta_n) \to T_\phi^\phi(\eta) \) weakly). Let \( \eta_n \to \eta \) weakly. Then by stochastic continuity of \( \lambda_j(\cdot|s) \) we obtain that \( s' \to \lambda_j([s, \bar{S}]|s') \) is continuous. Therefore,

\[
\int_S \lambda_j([s, \bar{S}]|s') \eta_n(ds') \to \int_S \lambda_j([s, \bar{S}]|s') \eta(ds')
\]

for all \( s \). This implies that \( T_\phi^\phi(\eta_n) \to T_\phi^\phi(\eta) \) weakly.

Let \( \eta^*_t \) be a distribution of \( s_t^\phi \) and \( \eta^*_t = \delta_{\bar{s}} \). Then by previous step \( \eta_t \) is stochastically decreasing. Hence it is weakly convergent to some \( \eta^* \). Therefore and by continuity of \( T^\phi \) we have \( \eta^* = T_\phi^\phi(\eta^*) \). By definition of

\(^{14}\)It means that \( s' \to \lambda_j([s, \bar{S}]|s') \) is continuous.
we immediately obtain $\eta(\phi^*) \preceq \eta^*$. Note that by stochastic monotonicity of $T_{\phi^*}(\cdot)$ we recursively obtain that $\delta S \geq \eta(\phi^*)$ and hence $\eta^* \geq \eta(\phi^*)$. As a result $\eta^* = \eta(\phi^*)$. Similarly we show convergence of the sequence of distributions $\delta_t \phi^*$.

Firstly observe that our result in stronger than theorem 5.2 stated in Curtat (1996). Not only we do characterize sets of invariant distributions associated with extremal strategies but also prove week convergence of distributions to the greatest invariant one. Here we have to comment on the greatest invariant distribution induced by any SMNE. If for almost all $s \in S$ we have $\sum_j g_j(s, \cdot) < 1$ we obtain a positive probability of reaching zero (an absorbing state) each period and hence the only invariant distribution is delta Dirac at zero. Hence to obtain a nontrivial invariant distribution one have to assume $\sum_j g_j(s, \cdot) = 1$ for all $s$ in some subset of a state space $S$ with positive measure, e.g. interval $[S', \bar{S}] \subset S$ (see Hopenhayn and Prescott (1992) or more recently Kamihigashi and Stachurski (2010)).

Secondly let us mention that theorem 4.1 and 4.2 can also imply a monotone comparative dynamics (as defined by Huggett (2003)) with parameter $\theta \in \Theta$ results implied by extremal SMNE: $\phi^*, \psi^*$. For this define the greatest invariant distribution $\eta_\theta(\phi^*_\theta)$ induced by $Q(\cdot | s, \phi^*_\theta, \theta)$ and greatest invariant distribution $\eta_\theta(\psi^*_\theta)$ induced by $Q(\cdot | s, \psi^*_\theta, \theta)$.

**Corollary 4.1** Let assumptions of theorem 4.2 be satisfied. Then $\eta_{\theta_2}(\phi^*_\theta_2) \succeq \eta_{\theta_1}(\phi^*_\theta_1)$ as well as $\eta_{\theta_2}(\psi^*_\theta_2) \succeq \eta_{\theta_1}(\psi^*_\theta_1)$ for any $\theta_2 \geq \theta_1$.

**Proof:** By theorem 4.2 we conclude existence of the greatest fixed point of $T_{\phi^*}^{\theta_2}$ and $T_{\phi^*}^{\theta_1}$. We also observe that $T_{\phi^*}^{\theta}$ is weakly continuous. Then we notice that $\theta \rightarrow T_{\phi^*}^{\theta}$ is increasing under first stochastic dominance order on the chain complete poset of probability measures on the compact state set.

Consider a sequence of iterations from a $\delta S$ generated on $T_{\phi^*}^{\theta}$ and observe that by Kantorvich-Tarski theorem (Dugundji and Granas (1982), theorem 4.2) we have $\sup_n T_{\phi^*}^{n, \theta_2} = \eta_{\theta_2}$ and similarly $\sup_n T_{\phi^*}^{n, \theta_1} = \eta_{\theta_1}$. Now as for any $n$ we have $T_{\phi^*}^{n, \theta_2} \succeq T_{\phi^*}^{n, \theta_1}$ by the above weak continuity we obtain that:

$$\eta_{\theta_2} = \sup_n T_{\phi^*}^{n, \theta_2} \succeq \sup_n T_{\phi^*}^{n, \theta_1} = \eta_{\theta_1}.$$  

Similarly we consider $\psi^*$.

Finally let us stress that by weak continuity of operators $T_{\phi^*}$ and $T_{\psi^*}$, we can obtain results allowing to estimate parameters $\theta$ of our stochastic game using simulated moments methods (see Pakes and Pollard (1989) and Lee and Ingram (1991)) or more recently Aguirregabiria and Mira (2007).
5 Sequential equilibrium via KP/APS methods in Function Spaces

We now consider an KP/APS procedure for constructing all measurable sequential (Markov) Nash equilibrium for our infinite horizon stochastic game. Our approach is distinct from those taken in the existing literature as we operate directly in function spaces. In particular, we construct a monotone operator in the powerset of the function spaces of bounded measurable maps. By working directly with function spaces, we avoid conceptual problems when working with correspondences that simply admit measurable selections (where, for example, procedures for constructing, characterizing, and approximating such selections can prove to be very difficult to address).

In our approach to KP/APS method, we construct decreasing subsets of bounded measurable functions using an operator that maps subsets of function spaces into themselves (order with set inclusion) whose limit encloses the set of sequential Markov Nash equilibria. The convergence structure, in general, is studied using notions of weak* topology; in some cases (e.g., the case of Lipschitz continuous sequential equilibrium), we can strengthen our notion of convergence to uniform convergence.

In motivating the importance of our results, it bears mentioning that for dynamic games with more restrictive shocks spaces (e.g., discrete or countable), KP/APS procedures has been used extensively in economics in recent years: e.g. by Kydland and Prescott (1980), Atkeson (1991), Pearce and Stacchetti (1997), Phelan and Stacchetti (2001) for policy games, and Feng, Miao, Peralta-Alva, and Santos (2009) for dynamic competitive equilibrium in recursive economies. Related methods have been proposed for theoretical work in stochastic games (e.g., Mertens and Parthasarathy (1987) and Chakrabarti (1999, 2003)). For our work, we follow the original Abreu, Pearce, and Stacchetti (1990) procedure, but only work with mappings between subsets of functions. For this end we require the following assumptions.

Assumption 6 We let:

- $u_i$ is continuous on $S \times A$, $u_i$ is bounded by $\bar{u}$,
- $u_i$ is supermodular in $a_i$ for any $s, a_{-i}$, and has increasing differences in $(a_i; a_{-i}, s)$,
- for all $s \in S$ the sets $\tilde{A}_i(s)$ are compact intervals and multifunction $\tilde{A}_i(\cdot)$ is upper hemicontinuous and expanding under set inclusion i.e. if $s_1 \leq s_2$ then $\tilde{A}_i(s_1) \subseteq \tilde{A}_i(s_2)$.

\footnote{For example, see the discussion in Phelan and Stacchetti (2001, p. 1500-1501), which discusses such possibility in function spaces. In this section, we show such a procedure is analytically tractable, and discuss it computational advantages in addition.}
\( Q(ds' | , \cdot) \) is stochastically continuous on \( S \times A \),

\( Q(ds' | s, a) \) is stochastically supermodular in \( a_i \) for any \( s, a_{-i} \) and has stochastically increasing differences in \( (a_i, a_{-i}, s) \),

\( Q(ds' | s, a) \) has density \( q(s' | s, a) \) with respect to some \( \sigma \) finite measure \( \mu \) i.e. \( Q(A | s, a) = \int_A q(s' | s, a) \mu(ds') \). Assume that \( q(s' | s, \cdot) \) is continuous and bounded for all \( (s', s) \) and for all \( s \in S \)

\[ \int_S ||q(s' | s, \cdot)||_\infty \mu(ds') < \infty. \]

- support of \( Q \) is independent of \( (s, a) \).

Under these assumptions, as above by standard arguments (state dependent) sequential equilibria and corresponding set of values \( V^* \) can be analyzed using a static auxiliary (or super-) game \( \{1, \ldots, n\}, \{\hat{A}_i(s), \Pi_i\}_{i=1}^n \) where payoffs

\[ \Pi_i(v_i, s, a) := (1 - \beta_i) u_i(s, a) + \beta_i \int_S v_i(s') Q(ds' | s, a), \]

are parameterized by a Borel measurable, continuation values \( v = (v_1, v_2, \ldots, v_n) \), and the state \( s \in S \). With fixed \( v \) and \( s \) the game is called \( G_v^s \). If \( v \) is increasing function, then under these assumptions \( G_v^s \) is a supermodular game and hence (see Milgrom and Roberts (1990) or Zhou (1994)) has a complete lattice of fixed points. By \( NE(v, s) \) we denote the set of Nash equilibria in \( G_v^s \).

Our version of the KP/APS procedure relative to the set of sequential equilibrium an infinite horizon stochastic game is the following. Consider a set of \( n \)-tuple Borel measurable, continuation values \( V \), with all coordinates bounded by 0 and \( \bar{u} \). Denote with \( \mathcal{V} \) the powerset of \( V \) ordered by set inclusion. For any subset \( V' \in \mathcal{V} \), define an operator \( B \) by

\[ B(W) = \bigcup_{v \in W} \{ w \in V' \in \mathcal{V} : w(s) = \Pi(v, s, a^*), a^* \in NE(v, s) \} \].

It is easy to see that \( B \) is increasing under set inclusion, and maps the space \( \mathcal{V} \) into \( \mathcal{V} \). Further, it is well known that \( \mathcal{V} \) is a complete lattice (actually, a continuous lattice). Finally, denote by \( V^* \in \mathcal{V} \) the set of equilibrium values corresponding to all sequential equilibria of our stochastic game.

We first prove a few important lemmas prior to proving our central theorem in this section. This result, namely theorem 5.1, shows that iterations on an operator \( B \) starting from some upper element \( V^U \in \mathcal{V} \) that is mapped down (under set inclusion) converges pointwise (in the Hausdorff metric
topology on $V$) to the greatest fixed point $W^*$ of $B(W)$. Further, in theorem 5.2, we show this greatest fixed point $W^*$ is a set of all sequential equilibria values, i.e. $W^* = V^*$. As a result, it is clear that our sequential Markov Nash Equilibrium computed in the previous section is an element of $W^*$.

We first prove a few important lemmata. Our first lemma considers the fixed point set of $G_{v_s}$.

**Lemma 5.1 (Complete lattice of $NE(v,s)$)** If $v \in V$ and $v$ is increasing, then set $NE(v,s)$ is a nonempty completely lattice.

**Proof:** Since $v$ is an increasing function, hence $\Pi_i(v_i,s,a)$ is supermodular in $a_i$ and has increasing differences in $(a_i,a_{-i})$. Hence the thesis follows directly from Milgrom and Roberts (1990).

We mention the following corollary of this lemma that we use in the sequel:

**Corollary 5.1** If $W$ contains increasing functions then $B(W) \neq \emptyset$, moreover $B(W)$ is also completely lattice.

**Lemma 5.2 (Convergence of values)** Let $v^t \rightarrow v$ in the weak* topology on $L_\infty(\mu)$ and $a^t \rightarrow a$ as $t \rightarrow \infty$. Then $\Pi(v^t,s,a^t) \rightarrow \Pi(v,s,a)$ pointwise in $s \in S$.

**Proof:** By assumptions 6 we just need to show that

$$\int_S v_i^t(s')q(s')\mu(ds') \rightarrow \int_S v_i(s)q(s|s,a)\mu(ds').$$

Indeed

$$\left|\int_S v_i^t(s')q(s')\mu(ds') - \int_S v_i(s)q(s|s,a)\mu(ds')\right|$$

$$\leq \left|\int_S v_i^t(s')q(s')\mu(ds') - \int_S v_i^t(s')q(s'|s,a)\mu(ds')\right|$$

$$+ \left|\int_S v_i^t(s')q(s'|s,a)\mu(ds') - \int_S v_i(s)q(s'|s,a)\mu(ds')\right|$$

$$\leq a\int_S |q(s') - q(s|s,a)|\mu(ds')$$

$$+ \int_S v_i^t(s')q(s'|s,a)\mu(ds') - \int_S v_i(s)q(s'|s,a)\mu(ds')$$

$$\leq \bar{a}$$

$$+ \int_S v_i^t(s')q(s'|s,a)\mu(ds') - \int_S v_i(s)q(s'|s,a)\mu(ds')$$

(2)
The last term converge to 0 by definition of weak*-convergence. We show that the term in (2) converge to 0. By Assumption 6 the function from integral converge to 0. Note that $|q(s') - q(s')| \leq 2||q(s'|s,\cdot)||_\infty$. Since by Assumption 6, $||q(s'|s,\cdot)||_\infty$ is integrable function, the convergence of integrals follows from Lebesgue Dominance Theorem.

Now consider a sequence of subsets of equilibrium values $\{W_t\}_{t=1}^\infty$ that are generated by iterations on our operator $B(W)$ from some initial subset $W_1 = V$ (with $W_{t+1} = B(W_t)$). In the end, we produce our version of the typical convergence result for KP/APS procedures, only now the set convergence structure is modeled using the induced topology of the function space $L_\infty(\mu)$. We need a last lemma before proving our central result of this section concerning the compactness of subsets $W_t$ in $L_\infty(\mu)$.

**Lemma 5.3 (Compactness of $W_t$)** For each $t \in \mathbb{N}$, $W_t$ is compact set in the weak*-topology on $L_\infty(\mu)$.

**Proof:** Since $V$ is a set of functions with the image in $[0, \bar{u}]$, hence, by Alaoglu theorem it is weak*-compact set. Now we show that $W_t$ is compact for $t > 1$. To do it is sufficient to show that $B(W)$ is weak*-compact whenever $W$ is.

Let $w^t \in B(W)$ for all $t$. Then, by definition of $B(W)$, we have $w^t(s) = \Pi(v^t, s, a^t(v^t, s))$, where $v_t \in W$. As both $W$ and $V$ are compact, without loss of generality, assume $v^t \to v^*$ where $v^* \in W$, and $w^t \to w^*$ (where convergence in both cases is in the weak*-topology). Fix $s > 0$. Then, $a^t(v^t, s) \to a^*$. We now show that $a^*$ is Nash equilibrium in the reduced game $G_{v^*}$. By Lemma 5.2, $\Pi(v^t, s, a^t) \to \Pi(v^*, s, a^*)$; hence, $a^*$ is a Nash equilibrium in the static game $G_{v^*}$. For each $s$, we can define $a^*(s)$ as a Nash equilibrium function\(^{16}\). Therefore, we have $w^* := \Pi(v^*, s, a^*(v^*, s))$, $w^* \in B(W)$ and $w^*$ is a weak*-limit of the sequence $w_t$.

We are now ready to prove our first central computation of equilibrium result of this section. In particular, we construct the greatest fixed point of $B(W)$ by successive approximations in $V$

**Theorem 5.1** Operator $B$ has the greatest fixed point $W^*$ and $W^* = \lim_{t \to \infty} W_t = \bigcap_{t=1}^\infty W_t$.

\(^{16}\)It is not clear that $a^*$ is measurable, but for us it is enough to obtain measurability of $w^*$.  

28
Lemma 5.4 (Self generation)

Let $V$ be the set of all player 1's strategies. For each player $i$, define $V_i$ as the set of all strategies $v_i^t = \Pi_i(v^t_i, s, a^t)$. Let $V^\infty := \lim_{t\to\infty} V_i$. We need to show that $V^\infty = V^*$. Clearly, $V^\infty \subset W_t$ for all $t \in \mathbb{N}$, hence

$$B(V^\infty) = B\left(\bigcap_{t=1}^{\infty} W_t\right) \subset \bigcap_{t=1}^{\infty} B(W_t) = \bigcap_{t=1}^{\infty} W_t + 1 = V^\infty.$$ 

To show equality, it suffices to show $V^\infty \subset B(V^\infty)$. Let $w \in V^\infty$. Then, $w \in W_t$ for all $t$. By the definition of $W_t$ and $B$, we obtain existence of the sequence $v^t \in W_t$ and Nash equilibrium $a^t$ such that

$$w(s) = \Pi(v^t, s, a^t).$$

Since $V$ is compact, without loss of generality, assume that $v^t$ weakly converge to $v^*$. Moreover, $v^* \in V^\infty$, since $W_t$ is a descending set of compact sequences. Fix arbitrary $s > 0$. Without loss of generality, let $a^t \to a^*$, where $a^*$ is some point from $A$. We show that $a^*$ is a Nash equilibrium in the static game $G^*_s$. Let $a_i \in A_i$. Then, for some $\tau \in \mathbb{N}$:

$$\Pi_i(v^*_i, s, a^\tau) \geq \Pi_i(v^*_i, s, a^\tau, a_i).$$

By lemma 5.2, if we take a limit in this expression, we obtain $a^*$ a Nash equilibrium in the static game $G^*_s$, and $w(s) = \Pi(v^*, s, a^*)$. We obtain $w \in B(V^\infty)$. Hence, $V^\infty$ is a fixed point of $B$, and, by definition $V^\infty \subset W^*$. To finish the proof, we need to show that $W^* \subset V^\infty$. Since $W^* \subset V$, $W^* = B(W^*) \subset B(V) = W_1$. By induction, we can easily prove that $W^* \subset W_t$ for all $t$; hence, $W^* \subset V^\infty$. Therefore, $W^* = V^\infty$, which completes the proof. 

With the existence of $W^*$ guaranteed, as in KP/APS, we need to next show the so-called self generating property of $V^*$ in our setting. We do this in the following lemma:

**Lemma 5.4 (Self generation)** If $W \subset B(W)$ then $B(W) \subset V^*$. 

**Proof:** Let $w \in B(W)$. Then, we have $w = \Pi(v^1, s, \sigma^1(s))$ for some $v^1 \in B(W)$ and Nash equilibrium $\sigma^1(s) \in NE(v_1, s)$. Then, we have $v_1 \in W$ by self generating property $v_1 \in B(W)$. Consequently, we can define $v_{t+1} = \Pi(v_t, s, \sigma^t(s))$ and $\sigma^t \in NE(v_t, s)$, all $v_i \in W$. Clearly, the Markovian strategy $\sigma$ generates payoff vector $w$. We next need to show this is a Nash equilibrium in the stochastic game for ($\mu$-almost) all initial states. Suppose that only player $i$ use other strategy $\tilde{\sigma}_i$. Then, for all $t$, we have $v_{t+1} = \Pi(v_t, s, \sigma^t(s)) \geq \Pi(v_t, s, \sigma^t(s), \tilde{\sigma}_i)$. If we take a $T$ truncation
\[ \sigma^{T,\infty} = (\tilde{\sigma}^1, \ldots, \tilde{\sigma}^T, \sigma^{T+1}, \sigma^{T+2}, \ldots) \]

this strategy can not improve a payoff for player \( i \). Indeed,

\[ U_i(\sigma, s) \geq U_i(\sigma_{-i}, \sigma_i^{T,\infty}, s) \rightarrow U_i(\sigma_{-i}, \tilde{\sigma}_i, s) \]

as \( T \rightarrow \infty \). As \( u_i \) is bounded , and the queue of the sum \( U_i(\sigma_{-i}, \sigma_i^{T,\infty}, s) \) depending on \( \sigma_i \) is bounded by \( \bar{u} \), we multiply by \( \beta_T^i \).

Theorem 5.2 (Sequential NE value set is the greatest fixed point of \( B \))

We have \( V^* = W^* \).

Proof: We show that \( V^* \) is a fixed point of operator \( B \). Clearly \( B(V^*) \subset V^* \). So we just need to show the reverse. Let \( v \in V^* \) and \( \sigma = (\sigma_1, \sigma_2, \ldots) \) be a profile supporting \( v \). By assumption 6 \( \sigma_{2,\infty} = (\sigma_2, \sigma_3, \ldots) \) must by a Nash equilibrium \( \mu \) almost everywhere i.e. a set of initial states \( S_0 \) which \( \sigma_{2,\infty} \) is not a Nash equilibrium must satisfy \( \mu(S_0) = 0 \). We could define \( \tilde{\sigma}(s) = \sigma_{2,\infty} \) for \( s \not\in S_0 \) and \( \tilde{\sigma}(s) = \sigma \) if \( s \in S_0 \). Let \( \tilde{v} \) be equilibrium payoff generated by \( \tilde{\sigma} \). Clearly \( \tilde{v} \) is \( \mu \) measurable. Hence \( v(s) = \Pi(\tilde{v}, s, \sigma_1) \) and hence \( v \in B(V^*) \) and \( V^* \subset B(V^*) \). As a result \( B(V^*) = V^* \).

Finally by definition (greatest fixed point) of \( W^* \) we conclude that \( V^* \subset W^* \). To obtain the reverse inclusion we apply lemma 5.4. Indeed \( W^* \subset B(W^*) \), hence \( W^* \subset V^* \) and we obtain that \( V^* = W^* \).

Few comments are in order. First, for strategic dynamic programming ala KP/APS, the method require that the auxiliary game has a Nash equilibrium. In our methods, to guarantee existence in this auxiliary game, we assume its supermodular structure. This assumptions can be weakened. For example Chakrabarti (2003) studies the existence of stationary, Markovian equilibria using similar techniques but for a (nonatomic) measure of players. This assumption allows him to obtain results on pure strategy equilibria. See also Chakrabarti (2008) for another, concavity based approach to deal with existence of a Nash equilibrium in pure strategies. In our paper, for an easy comparison with the result of monotone operator method presented in the previous section, we keep this specification. Secondly the assumption of the full (or invariant) support is critical for this "recursive type" method to work. That is having a sequential selection of values from \( V^* \) we can generate supporting sequential (time and state dependent) Nash equilibrium. Thirdly observe that the APS procedure does not imply that the (Bellman type) equation \( B(V^*) = V^* \) is satisfied for a particular value function \( v^* \) but only by a set of value functions \( V^* \). Hence existence of a stationary equilibrium cannot be deduced. Finally observe that the characterization of a equilibrium strategy in a particular period is week, i.e. all we can conclude is that it is a measurable function. Also this type of procedure does not offer
a numerical approximation of sequential NE or its value. An exception in terms of numerical characterization of a sequential NE value set is offered by Judd, Yeltekin, and Conklin (2003).

6 Applications

Application of our theorems are immediate and can be used for the games studied by Curtat (1996), Amir (2002), Horst (2005) or Nowak (2007) like dynamic (price or quantity) oligopolistic competition or “week” social interactions. We now discuss, however, two other possible applications of our results. First we show how our existence proof from section 3 can be used to show existence of a Markov equilibrium of dynamic oligopoly. Second we discuss comparative statics and equilibrium dynamics in the dynamic search model with learning.

6.1 Markov perfect industry dynamics

Consider $N$ firms (both incumbents or entrants) engaged in an infinite horizon, dynamic competition in an oligopolistic industry with investment, entry, and exit a la Ericson and Pakes (1995) or Doraszelski and Satterthwaite (2010). Let $S = ([0, 1] \times [0, \overline{w}])^N$ be $2N$ dimensional vector specifying for each period state $s_i = (\eta_i, w_i)$ of each firm, where $\eta_i$ is a probability of a firm being outside the market (i.e. $\eta_i = 0$ if firm $i$ is an incumbent and $\eta_i = 1$ if firm is a possible entrant), while $w_i \in [0, \overline{w}_i] \subset \mathbb{R}_+$ its individual productivity shock. Each period an action of a firm is $a_i = (p_i^{\text{exit}}, p_i^{\text{stayout}}, x_i) \in [0, 1]^2 \times [0, X_i]$ specifying (for incumbent) probability of exiting the market; (for possible entrant) probability of not entering the market; and finally (for incumbent) an investment decision. Given state $s$ and action profile $a$ the within period payoff of firm $i$ is given by:

$$u_i(s, a_i, a_{-i}) = (1 - \eta_i)[\pi_i(s) - \phi_i(s)p_i^{\text{exit}} - (1 - p_i^{\text{exit}})c_i(x_i, s)] +$$
$$+ \eta_i[(1 - \eta_i)(-\overline{\phi}_i(s) - c_i(x_i, s))],$$

where $\pi_i$ is a function specifying (for each state) within period incumbent’s profit from an oligopolistic market (depending on a particular example); $\phi_i$ firm’s scrap value after exit; $\overline{\phi}_i$ firm’s sunk entry costs while $c_i$ cost of investing $x_i$ units. The transition probability on $S$ is given by $Q$ and summarizes particular assumption of a model.

Let $\pi_i, \phi_i, \overline{\phi}_i$ be measurable while $c_i$ continuous in the first variable and measurable in the second. Normalize $c_i(x_i, 0) = 0$, $\phi_i(0) = 0$ and $\pi_i(0) = 0$. Further assume that $Q$ is given by assumption 2. Then we conclude from theorem 3.1 existence of a measurable Markov perfect Nash equilibrium. Moreover approximation procedure and bounds from theorems 3.1 and 3.2 hold.
Observe that supermodularity assumptions on \( u_i \) are naturally satisfied by appropriate choice of decision variables. Also as \( u_i \) does not depend on current competitors’ (actual in the market nor possible entrants) decisions other assumption of theorem 3.1 are satisfied by definition.

Concerning assumption placed on \( Q \). The 0 absorbing state assumption means that there is a probability that our market would be widely opened, all firms will enter and drive down economic profit to zero. Concentrating on \( g_j \) observe that required monotonicity in actions \((p^{exit}, p^{stayout}, x)\) has natural economic applications: higher probability of exit or staying outside market increases next period probability (of a drawn from \( \lambda_j \)) of being outside a market and higher investment increases next period probability of high state \( w \). Similarly supermodularity in actions means that the higher investment (or exit / stayout) decisions of a particular firm the higher the ”marginal probability” of others firms receiving high state (drawn from \( \lambda_j \)).

A careful reader notices that our theorems allows for much broader specification of a Ericson and Pakes (1995) or Doraszelski and Satterthwaite (2010) game; allowing e.g. current payoffs depend on others current actions or discrete decisions (see remark 1). Finally let us stress that such general existence results cannot be possible obtained from theoretical papers of Ericson and Pakes (1995), Curtat (1996), Amir (2002, 2005), Nowak (2007) or Doraszelski and Satterthwaite (2010). Also applied papers by Amir and Lambson (2003) are (abstracting from our transition specification) a special cases of our game as e.g. they require monotonicity of \( \pi_i \) with a state variable (number of active firms).

Also our natural approximation procedure allows further applied researcher in the field including parameters estimation and strategies computation.

### 6.2 Dynamic search with learning

We extend a dynamic game of search\(^{17}\) with learning studied by Curtat (1996), Amir (2002), Horst (2005) or recently Nowak (2007). Each period \( N \) traders (players) expands effort searching for trading partners (other players). Denoting search effort level by \( a_i \in [0, s] = A_i(s) \) and search costs by \( c_i(a_i) \) we let state \( s \in S = [0, \bar{S}] \subset \mathbb{R}_+ \) represents the current productivity level of the search process, with a current search reward given by:

\[
u_i(s, a) = sf(a_i, a_{-i}) - (1 - \theta)c_i(a_i),\]

where \( f \) is a ”matching” function and \( \theta \in [0, 1] \) a search subsidy. Let \( s \) be drawn from a transition \( Q \).

Let \( f, c_i \) be continuous and increasing with \( f \) supermodular. Additionally, similarly to Curtat (1996), Horst (2005) or Nowak (2007), let \( Q \) be given

\(^{17}\)Originating from Diamond (1982) search model.
by assumption 5. Then theorem 4.1 implies existence of extremal Markov perfect Nash equilibria, each monotone with productivity $s$ and subsidy $\theta$. Also theorem 4.2 and corollary 4.1 imply existence of extremal invariant distributions monotone with subsidy $\theta$. This way we also present for the first time, to the best of our knowledge, general multiplicity of (extremal) ordered equilibria and invariant distributions possibility, counterparting Diamond (1982) results on inefficient steady states and suitable stabilization policies. Let us finally stress that as observed by Nowak (2007) the least equilibrium may be trivial for a common (additive) specification of $f$. This example stresses again the advantage of our constructive methods over results of Curtat (1996), Amir (2002) or Horst (2005). As the trivial equilibrium can be constructed the question of "general" existence results (addressed using topological fixed point theorems) remains moot.

7 Conclusion

Our constructive approach in this paper leads to the possibility of more fundamental explorations into characterizing the structure of iterative procedures indexed on the natural numbers for computing and characterizing SMNE in stochastic games. That is, given the central role of order continuity in this work, it points to the need for a deeper understanding of the relationship between order and topology in the structure of the methods proposed. We are pursuing such a theory of computation in current work. In particular, we are seeking sufficient conditions under which we can develop a continuous theory of computation for stochastic games with strategic complementarities (e.g., see Balbus, Reffett, and Woźni (2010a) for preliminary discussions). Aside from considering such questions for games with stochastic transitions with mixing assumptions on the noise, we are also pursuing a related question of when stochastic games with stochastic transitions parameterized by such noise can provide approximations to SMNE in stochastic games with more general transition structures.

Along these lines, following the seminal work in continuous lattices of Scott (1970, 1972), first explore the relationship between order topologies and uniform topologies in Scott continuous operators in complete lattices of continuous functions. Making extensive use of recent developments in domain theory and continuous posets/lattices in computer science as applied to stochastic games, it turns out the theory of approximation produced by the Scott topology in this context (with a countable or uncountable basis) appears to be the appropriate setting for obtaining such a unified theory of the continuous computation in stochastic games with strategic complementarities. To understand the issues at hand, we first mention it is known that for computational problems in complete lattices, say $X$, the mapping $f : X \to X$ is order continuous iff $f$ is Scott continuous (i.e., for all direct
sets $X' \subset X$, $\vee f(X') = f(\vee X')$}. Many interesting equilibrium problems in stochastic games (and, more generally, economics) take place in the setting of complete lattices of continuous functions (relative to pointwise orders). Additionally, for KP/APS procedures, computational problems take place in complete lattices of powersets ordered under set inclusion (or, perhaps, reverse set inclusion). Each of these are continuous lattices, or actually, completely distributive complete lattices (henceforth CDCLs). Further, if we consider all the non-empty order intervals of continuous functions ordered under Veinott’s strong set order, this turns out to also be a CDCL. Hence, these are all suitable domains to develop a theory of computation using Scott continuity.

In Gierz, et. al. 2003, numerous characterizations of order continuity are provided, as well as situations where order continuity is necessary to obtain constructive theories of computation without appeals to transfinite induction, generalized iteration, or iterations indexed in the ordinals. Therefore, to understand the foundations of constructive methods in stochastic games (or, more generally, classes of games with strategic complementarities such as superextremal/quasisupermodular games), it turns out what appears to be needed are action spaces that are CDCLs. As many problems in economics with strategic complementarities have precisely this structure (e.g., as set in complete lattices that are order isomorphic to some power of the unit interval), such constructive results could find general application in economics.

For a deeper understanding of the conditions on the primitive data of stochastic games needed to deliver a continuous theory of computation of SMNE (in either pointwise partial orders or set inclusion partial order), it turns out the role of Scott continuity in these constructions is central. In continuous lattices, a rigorous theory of approximation is available. Understanding what classes of games (stochastic games, or not) deliver this sort of topological continuity turns out to be central to the question of constructive methods. (e.g., see Balbus, Reffett, and Woźny (2010b)). Take, for example, the fixed point set in games where under pointwise partial orders, equilibria are $\sigma$–complete lattices. If we use equivalence classes, however, ala Vives and Van Zandt (2007) or Van Zandt (2010) for Nash equilibrium in Bayesian games, iterations take place in continuous complete lattices again. The differences in these two results is simply a matter of choice of representation for the Scott topology used to model continuity. This is true for one-shot games with strategic complementarities (i.e., Van Zandt’s one-shot supermodular game), or infinite horizon stochastic games, such as those in this paper.

To best understand the need for further research the illuminates the interplay between order continuity, choice of partial order, computation, and the Scott topology where convergence/approximation/continuity are mod-

\[^{18}\text{See, for example, Scott (1972), Proposition 2.5.}\]
eled, consider the case of our stochastic games under conditions where Lipschitz continuous SMNE exist in equicontinuous subset of continuous functions on the state space $S$, say $CM(S)$, where $S \subset \mathbb{R}^n$ (or, more generally, $S$ is an ordered Polish space). Under pointwise orders, $CM(S)$ is a CDCL. Further, the powerdomain $PI(CM(S))$ of all the (closed, hence compact) nonempty subintervals partially ordered with Veinott’s strong set order is also CDCL (as this function space is order isomorphic to a power of the unit interval, equicontinuity closes the continuity property inf and sup operations under pointwise orders, and the topology of pointwise convergence coincides with the topology of uniform convergence).\(^{19}\) Further, $PI(CM(S))$ space can again be viewed as being order isomorphic to $CM(S)$ under Veinott’s strong set order. Hence, the constructive results of our section 3 can be understood using Scott continuity in this setting (where the Scott topology is placed either on $CM(S)$, or its power domain $PI(CM(S))$). Of course, $PI(CM(S))$ is also a CDCL under set inclusion. This means we can equivalently pursue a theory of computation based upon our KP/APS in function spaces (which, in the case that our condition on the stochastic game imply that SMNE take place in $CM(S)$, provides a second view of computing SMNE for such games).

Where things become interesting, though, is when SMNE take place in function spaces which, at best, are countably chain complete and/or sigma-complete. Using KP/APS for such games, one arrives at a theory similar to that case of SMNE in $CM(S)$ (as powersets, or powerintervals ordered set inclusion remain continuous lattices). The point is for $\sigma$-complete lattices, powerdomains in ordered Veinott strong set order (in induced pointwise orders) are no long even lattices necessarily. If we work with equivalence classes (with a.e. partial orders), we can again recover the continuous lattice structure of the set. But then we have weaken the requirements for completeness (and continuity of the lattice), and hence, changed the implications of order continuity.

So our work in this paper suggests a further exploration of the relationship between order/Scott continuity, computation, and the choice of partial order in the theory of computation. It bears mentioning, that when order structures are not induced (on powersets), the nature of equilibrium comparative statics results change, both with respect to structure, computability, and continuity. These are the questions we shall pursue in our future work.

\(^{19}\)Let $X$ be a lattice, $P(X)$ be the “powerdomain” consisting of the nonempty sublattices of $X$ ordered Veinott strong set order, $2^X$ the powersets of $X$. Then, $P(X)$ is the largest poset in $2^X$ ordered Veinott strong set order, with $PI(X) \subset P(X)$ also poset.

Now, $P(X)$ is not necessarily a lattice (e.g., let $X$ be nondistributive). If $X$ is CDCL, $P(X)$ is a complete lattice; actually, it is a CDCL (hence, a continuous lattice). Further: $PI(X) \subset P(X)$ is a CDCL sublattice of $P(X)$. See Balbus, Reffett, and Woźni (2010a) for a discussion.

It is well-known that $2^X$ is a CDCL lattice under set inclusion (dually, a CDCL under reverse set inclusion).
References


——— (2010b): “Fixed point correspondences for Scott continuous operators on countably complete ordered sets with applications,” MS.


40


