Basic auction theory revisited*

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August 9, 2010

Abstract

We consider a two good world where an individual $i$ with income $m_i$ has utility function $u(x, y)$, where $x \in [0, \infty)$ and $y \in \{0, 1\}$. We first derive the valuation for good $y$ as a function of his income. Then we consider the following problem. Suppose good $x$ is available in a store at a fixed price 1. Good $y$ can be obtained in an auction. In such a situation we show that bidding ones own valuation is an equilibrium in a second-price auction. With risk neutral bidders and high enough incomes we derive the symmetric equilibrium in first-price and all-pay auctions and show that revenue equivalence fails to hold. With risk neutrality we also show that under mild restrictions, the revenue maximising reserve price is zero for all the three auctions and the all-pay auction with zero reserve price fetches the highest expected revenue. With low enough incomes, we show that under some restrictions, bidding ones own valuation is a symmetric equilibrium even for first-price and all-pay auctions. Here also, the expected revenue is the highest with all-pay auctions. Lastly, we provide sufficient conditions for the existence of symmetric equilibrium in first-price and all-pay auctions when bidders are risk averse with high enough incomes.

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*I am indebted to Claudio Mezzetti and Anjan Mukherji for this paper. Comments from Masaki Aoyagi, Marcus Berliant, Wen Chern, Hikmet Gunay, Junichiro Ishida, Noriaki Matsushima, Eiichi Miyagawa, Yasuyuki Miyahara, Ryo Ogawa, Shigehiro Serizawa, Makoto Yano and Lex Zhao were very helpful. The first draft of the paper was written when I was a ‘Visiting Foreign Scholar’ at the Institute of Social and Economic Research, Osaka University in 2010. ISER provided me with excellent research facilities and stimulating intellectual ambience and I am grateful for that. The usual disclaimer applies.
1 Introduction

In the traditional benchmark model of auctions we have the following scenario. There is one indivisible object up for sale and there are \( n \) potential bidders. Each bidder’s valuation (maximum price that he is willing to pay for the object), \( v_i \), is private information. Bidder \( i \) knows his (her) own valuation but does not know others’ valuations. He (she) only knows that \( \forall j \neq i, V_j \) lies in the interval \([\underline{v}, \overline{v}]\) with a known distribution function\(^1\). Seller also does not know any bidder’s valuation. He only knows that for all \( i \), \( V_i \) lies in the interval \([\underline{v}, \overline{v}]\) with a known distribution function. In a standard auction the object is sold to the highest bidder. The payment by each bidder depends on the type of auction used by the seller. There is a huge literature around this model\(^2\). One of the most celebrated results is the revenue equivalence theorem which states that under under certain assumptions (private values, independent types, symmetry, risk neutrality and no budget constraint), the expected revenue to the seller is same across a large class of auctions.

It may be noted that the independent private value model is based on quasilinear utility functions with high enough incomes. In such a case, valuations are independent of incomes. Consequently, private information about valuations tantamounts to private information about utility functions\(^3\).

In this paper we consider a more general class of utility functions. We consider a two good world where an individual \( i \) has utility function \( u(x, y) \), where \( x \in [0, \infty) \) and \( y \in \{0, 1\} \). The individual’s income is \( m_i \). This \( m_i \) may be thought of as the total amount of resources (or wealth) available to individual \( i \). Under standard assumptions we derive valuation, \( v(m_i) \), for good \( y \) as a function of his income. This links the budget constraint with valuation. We show that bidder’s valuation can never exceed his income. We also show that depending on the nature of the utility function and the income level, the valuation can be strictly increasing, constant or even strictly decreasing in income.

We look at the auction problem from a different angle. We assume that all individuals have the same utility functions but have different incomes. We treat incomes as types. Each individual’s income is private information and this implies that valuations (that are functions of income) are also private information. Then we consider the following problem. Suppose good \( x \) is available in a

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\(^1\) As per convention, we use capital letters to denote random variables and corresponding small letters to denote the realised values of such variables.

\(^2\) See Krishna (2010) for a very succinct survey

\(^3\) It does not matter whether incomes are private information or not, as long as they are high enough.
store at a fixed price 1. One unit of good \( y \) is sold at an auction house. The price of good \( y \) and the
winner will be determined by the Bayesian-Nash equilibrium of the auction game. Any individual
has the option of not participating in the auction. If he does not participate in the auction for
good \( y \), he spends his entire income on good \( x \) and earns utility \( u(m_i, 0) \). Consequently, in any
equilibrium the expected payoff to the individual will be at least \( u(m_i, 0) \). In such a model we
analyse first-price, second-price and all-pay auctions\(^4\).

In a second price auction we first show that for any given utility function, choosing a bid equal
to valuation is a weakly dominant strategy for any bidder \( i \) with income \( m_i \). This shows that the
original Vickrey (1961) result is very robust to changes in the benchmark model.

Unlike the second price auction, we do not have a general equilibrium existence result for first-
price and all-pay auctions for all possible income levels. As such, to analyse such auctions we need
to classify incomes into two categories. In our model we will define a critical level of income \( k \). We
later show that for any given utility function \( u(.) \), \( k \) is such that \( m_i \leq k \iff u(0, 1) \geq u(m_i, 0) \). If
all incomes are below \( k \) we say that incomes are low enough and if all incomes are above \( k \) we say
that incomes are high enough.

With risk neutral bidders and high enough incomes we derive the symmetric equilibrium in first-
price and all-pay auctions and show that revenue equivalence fails to hold. More specifically, we
show that the expected revenue is same with first price and second price auctions. However, all-pay
auction fetches strictly higher expected revenue than first price auction. With risk neutrality we
also show that under mild restrictions, the revenue maximising reserve price is zero for all the three
auctions. This stands in contrast to the benchmark model where the revenue maximising reserve
price is always positive. Moreover, among the three auctions analysed, the expected revenue to
the seller is highest in the all-pay auction with zero reserve price. Whether this is an optimal
mechanism or not remains an open question.

With low enough incomes, we show that with some restrictions, bidding ones own valuation is
a symmetric equilibrium even for first-price and all-pay auctions. Here the expected revenue is the
highest with all-pay auctions and lowest with second price auctions. We illustrate our results with
a specific numerical examples.

Lastly, we extend our results and provide sufficient conditions for the existence of symmet-

\(^4\)As usual, one can show that first-price auction is outcome equivalent to Dutch auction and second-price auction
is outcome equivalent to English auction.
ric equilibrium in first-price and all-pay auctions when bidders are risk averse with high enough incomes. The expected revenue ranking for this case remains an open research problem.

The plan of the paper is as follows. In section 2 we provide the model of our exercise. Section 3 provides the equilibrium in second-price auctions. In section 4 we analyse risk-neutral bidders with high enough incomes. Section 5 analyses equilibrium when incomes are low enough. In section 6, we provide some extensions of our model to the case when bidders are risk averse with high enough incomes. Lastly, the appendix gives the proofs of all the results.

2 The Model

We consider a two good world where an individual $i$ has utility function $u(x, y)$ where $x \in [0, \infty)$ and $y \in \{0, 1\}$. This means that any non-negative amount of good $x$ can be consumed but in case of good $y$ there are only two choices. Either one unit of good $y$ can be consumed or it cannot be consumed. Good $x$ may be thought of as a composite commodity. Individual $i$’s income is $m_i$.

We provide the first set (A) of assumptions. These assumptions impose some standard restrictions on the utility function.

Assumption A1. $u(0,0) = 0$ and for all $y \in \{0, 1\}$, $u(x, y)$ is continuous in $x$ for all $x \in [0, \infty)$ and partially differentiable w.r.t. $x$ for all $x \in (0, \infty)$.

Assumption A2. $u(x, y)$ is strictly increasing in both $x$ and $y$. That is, $u(x, 1) > u(x, 0)$ for all $x \in [0, \infty)$. And, for all $y \in \{0, 1\}$, $x_1 > x_2 \Leftrightarrow u(x_1, y) > u(x_2, y)$.

Assumption A3. $\lim_{x \to \infty} u(x, 0) > u(0, 1)$.

Note that assumptions A1- A3 imply that $u(0, 0) = 0 < u(m_i, 0) < u(m_i, 1)$ for all $m_i > 0$. The sign of $u(m_i, 0) - u(0, 1)$ is not known. Since $u(x, y)$ is continuous and strictly increasing in $x$ and since $\lim_{x \to \infty} u(x, 0) > u(0, 1)$ there exists a unique $k > 0$ s.t. $u(k, 0) = u(0, 1)$.

Hence

$$u(m_i, 0) \leq u(0, 1) \text{ iff } m_i \leq k$$
Let price of $x$, $P_x = 1$ and price of $y$ be $p$. Hence the individual’s problem is

$$\max_{x \in [0, \infty), y \in \{0,1\}} u(x, y)$$

s.t. $x + py \leq m_i$.

There are two possible cases. (i) $m_i \leq k$ and (ii) $m_i > k$.

We will now derive the valuation for good $y$ (the maximum price that the consumer is willing to pay for $y$) for each of these two cases.

**Case 1** $m_i \leq k \iff u(0,1) \geq u(m_i,0)$.

It may be noted that if $p \leq m_i$ and the individual purchases\(^5\) $y$ he can spend $m_i - p$ on $x$ and get $u(m_i - p, 1)$. If he does not purchase $y$ and spends his entire income on $x$ he gets $u(m_i, 0)$. Here note that for all $p \in (0, m_i]$

$$u(m_i, 1) > u(m_i - p, 1) \geq u(0,1) \geq u(m_i, 0)$$

Therefore the individual will buy good $y$ (that is choose $y = 1$) iff $m_i - p \geq 0 \iff p \leq m_i$. Hence, for this case the maximum price he is willing to pay for good $y$ (his valuation for $y$) is $v_i = m_i$.

**Case 2** $m_i > k \iff u(0,1) < u(m_i,0)$

From our assumptions it follows that for each $m_i$ there exists a unique $c(m_i)$ s.t. $u(c(m_i), 1) = u(m_i, 0)$. Clearly $c(.)$ is strictly increasing in $m_i$. As before, if $p \leq m_i$ and the individual purchases $y$ he can spend $m_i - p$ on $x$ and get $u(m_i - p, 1)$. If he does not purchase $y$ and spends his entire income on $x$ he gets $u(m_i, 0)$. Now

$$u(m_i - p, 1) \geq u(c(m_i), 1) = u(m_i, 0)$$

iff $m_i - p \geq c(m_i)$.

Therefore the individual will buy good $y$ (that is choose $y = 1$) iff $m_i - p \geq c(m_i) \iff p \leq m_i - c(m_i)$. That is, the maximum price he is willing to pay for good $y$ (his valuation for $y$) is

$v_i = m_i - c(m_i)$.

\(^5\)If $p > m_i$ then the individual cannot purchase $y$ and he has to spend the entire income on good $x$ and get $u(m_i, 0)$. 5
Therefore we get that valuations are functions of income.

\[ v(m_i) = \begin{cases} 
  m_i & \text{if } m_i \leq k \\
  m_i - c(m_i) & \text{if } m_i > k 
\end{cases} \]

where \( c(m_i) \) is such that \( u(c(m_i), 1) = u(m_i, 0) \).

**Comment** Note that when \( m_i \in [0, k] \) (income is low enough) the valuation \( v_i = m_i \) are strictly increasing in income. When \( m_i \in (k, \infty) \) (income is high enough) the valuation, \( v_i = m_i - c(m_i) \), may be increasing, constant or even decreasing in incomes. We will illustrate these with some examples.

**Example 1:** Suppose utility is quasilinear in \( x \). That is, \( u(x, y) = x + \hat{u}(y) \), where \( \hat{u}(1) > \hat{u}(0) = 0 \). Here \( u(0, 1) = \hat{u}(1) \) and \( u(m_i, 0) = m_i \).

Note that here \( k = \hat{u}(1) \) and valuation is

\[ v(m_i) = \begin{cases} 
  m_i & \text{if } m_i \leq k = \hat{u}(1) \\
  \hat{u}(1) & \text{if } m_i > k = \hat{u}(1) 
\end{cases} \]

Note that here valuation, \( v(m_i) \), is constant for all \( m_i > k \). In standard auction-theory the above utility function is used and the income is assumed to be high enough (i.e. \( m_i > k \)). Incomplete information about others’ valuation simply means incomplete information about \( \hat{u}(1) \). That is, for the standard independent private value model, each bidder knows his \( v_i = \hat{u}_i(1) \) but does not know other bidders \( v_j = \hat{u}_j(1) \). It may also be noted that since in the benchmark model for all \( i, m_i > k \), it does not matter whether incomes are private information or not (as valuations do not depend on incomes).

**Example 2.** Let \( u(x, y) = x(1 + y) + y \)

Note that here \( k = 1 \) and for \( m_i \in (1, \infty), c(m_i) = \frac{m_i - 1}{2} \). Hence

\[ v(m_i) = \begin{cases} 
  m_i & \text{if } m_i \leq k = 1 \\
  \frac{m_i + 1}{2} & \text{if } m_i > k = 1 
\end{cases} \]

Here \( v(m_i) \) is strictly increasing in \( m_i \) for all \( m_i \).
Example 3. Let $u(x, y) = x^2 + y$.

Note that here $k = 1$. Hence

$$v(m_i) = \begin{cases} 
  m_i & \text{if } m_i \leq k = 1 \\
  m_i - \sqrt{m_i^2 - 1} & \text{if } m_i > k = 1
\end{cases}$$

Here $v(m_i) = m_i - c(m_i) = m_i - \sqrt{m_i^2 - 1}$ is strictly decreasing for all $m_i \in (1, \infty)$.

2.0.1 Comment

It may be noted that for all cases $v_i \leq m_i$. While this seems trivial given the nature of our problem, it is important in the context of the literature. We draw attention to an interesting and influential paper by Che and Gale (1998) which drops the assumption of no budget constraint. A simplified version of their model is as follows. As before let $V_i = \text{bidder } i \text{'s valuation. But now, in addition, each bidder is subject to an absolute budget of } W_i \text{ and } V_i \text{ can be more than } W_i. \text{ In no circumstances can a bidder with a value-budget pair } (v_i, w_i) \text{ pay more than } w_i. \text{ If a bidder } i \text{ were to bid more than } w_i \text{ and default, then a penalty would be imposed. Each bidder's value-budget pair } (V_i, W_i) \text{ is identically and independently distributed on } [0, 1] \times [0, 1]. \text{ An important result of Che and Gale (1998) is that under certain conditions the expected revenue in the first price auction is higher than the expected revenue in the second price auction. It may be noted that in our set-up valuations are always less than or equal to income and consequently, we can avoid analysing the type of problem dealt with by Che and Gale (1998).}

2.1 Introducing incomplete information

2.1.1 The basic story

Our basic story is as follows. Let there be $n$ individuals. All have the same preference ordering (i.e. the same utility function). Bidder $i$'s income $M_i$ is private knowledge to the seller. Bidder $i$ knows that $M_i = m_i$ but does not know $M_j$ ($j \neq i$). Then private information about valuations tantamounts to private information about incomes. Suppose good $x$ is available at a store at price 1. One unit of good $y$ can be obtained in an auction. Price of good $y$ will be determined in the Bayesian-Nash equilibrium of the auction. Note that the consumer is no longer a price-taker. His bid affects the equilibrium price of $y$. 

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We now provide our next set (B) of assumptions.

Assumption B1. Independent types: $M_1,...,M_n$ are independently distributed.

Assumption B2. Symmetry: Each random variable $M_i \in [\alpha, \beta]$, where $\alpha \geq 0$, has the same distribution function $F(.)$ and associated strictly positive density $f(.)$. That is, each bidder $i$ believes that competitors’ incomes are given by $M_j \in [\alpha, \beta]$ with distribution function $F(.)$ and density function $f(.)$.

Assumption B3. In no circumstances can a bidder with income $m_i$ pay more than $m_i$ for any good.

Assumption B4. If a bidder $i$ were to bid more than $m_i$ (in the auction for good $y$) and default, then a penalty would be imposed.

Under assumptions A1-A3 and B1-B4 we will analyse three types of auctions viz (i) first-price auction (ii) second price auction and (iii) all-pay auctions.

Let $r$ be the reserve price. In any such auction the set of actions available for any bidder is $[r, \infty) \cup \{No\}$ where $\{No\}$ means that the bidder has the option of not participating in the auction. If he does not participate in the auction to buy good $y$, he simply spends all his income on good $x$ and earns utility $u(m_i,0)$. Individual $i$’s problem is to choose a strategy

$$b_i(m_i) : [\alpha, \beta] \longrightarrow [r, \infty) \cup \{No\}$$

so as to maximise

$$\text{Exp. } u(x,y)$$

$$\text{s.t. } b_i(m_i) \leq m_i \text{ if } b_i(m_i) \notin \{No\}.$$ 

Note that the expected payoff to an individual in any equilibrium is at least $u(m_i,0)$.

Before giving our main results we need to discuss some preliminaries on order statistics.
2.2 Order Statistics: some notations and preliminaries

Let $M_1, M_2..M_n$ denote a random sample of size $n$ drawn from $F(.)$. Then $M_{(1)} \geq M_{(2)} \geq \ldots \geq M_{(n)}$ where $M_{(i)}$s are $M_i$s arranged in decreasing magnitudes, are defined to be the order statistics corresponding to the random sample $M_1, M_2..M_n$.

We would be interested in $M_{(1)}$ (highest order statistic) and $M_{(2)}$ (second highest order statistic). The corresponding distribution functions and density functions are $F_1(.)$, $F_2(.)$ and $f_1(.)$, $f_2(.)$. Note that

\[
F_1(z) = F^n(z) \quad \text{and} \quad F_2(z) = F^n(z) + n \left[ F^{n-1}(z) - F^n(z) \right] \\
f_1(z) = nF^{n-1}(z) f(z) \quad \text{and} \quad f_2(z) = n(n-1) F^{n-2} [1 - F(z)] f(z)
\]

Another useful random variable is $T = \max \{M_2, M_3...M_n\}$. To any individual this is the maximum of the others’ incomes. The distribution and density function of $T$ is $G(.)$ and $g(.)$ respectively. Clearly

\[
G(t) = F^{n-1}(t) \\
g(t) = (n - 1) F^{n-2}(t) f(t)
\]

We will be using these notations frequently in our proofs.

3 Equilibrium in second price auctions

We first analyse second price auction. The result is identical to the standard result in auction theory.

**Proposition 1** If $r = 0$ then in a second-price auction for any $m_i \in [\alpha, \beta]$ it’s a weakly dominant strategy for bidder $i$ to bid his own valuation. That is, the following is a symmetric Bayesian Nash equilibrium of the second price auction.

\[
b^H (m_i) = v (m_i) = \begin{cases} 
  m_i & \text{if } m_i \leq k \\
  m_i - c(m_i) & \text{if } m_i > k
\end{cases}
\]
Comment  Irrespective of the nature of utility function and the level of incomes, bidding one's own valuation is a weakly dominant strategy in a second price auction. This shows that the original result of Vickrey (1961) on second price auction is very robust.

High enough and low enough incomes  We take $k$ as a critical benchmark of the level of incomes. If $\beta < k$ (that is, all incomes are below $k$) then we say that incomes are low enough. If $\alpha \geq k$ (that is, all incomes are above $k$) we say that incomes are high enough.

4  Risk neutral bidders with high enough incomes

We now consider the following class of utility functions where the individual is risk neutral. That is $\frac{\partial^2}{\partial x^2} u(x, y) = 0$. For an explanation about why this is so see Kreps (1990, chapter 3).

It is clear, that, to be consistent with our assumptions, we need a utility function of the following type.

$$ u(x, y) = \begin{cases} wx & \text{if } y = 0 \\ qx + s & \text{if } y = 1 \end{cases}, $$

where $q \geq w > 0$ and $s > 0$.

We now explain as to why we need $q \geq w > 0$ and $s > 0$. First note that if $q < w$ then for large enough $x$ we get $u(x, 0) = wx > qx + s = u(x, 1)$ and this violates assumption A2. If $s = 0$ then $u(0, 1) = 0 = u(0, 0)$ and this also violates assumption A2.

For this utility function $k = \frac{s}{w}$ and hence we have

$$ v(m_i) = \begin{cases} m_i & \text{if } m_i \leq \frac{s}{w} \\ \frac{(q-w)m_i+s}{q} & \text{if } m_i > \frac{s}{w} \end{cases} $$

Note that if $q = w$ then for all $m_i > \frac{s}{w}$ the valuations are constant and $v(m_i) = \frac{s}{q}$ and we are back to the standard auction case (similar to example 1).

Let $q > w$. This means valuations are strictly increasing in incomes for all $m_i$. And let $\alpha \geq k = \frac{s}{w}$. This implies that all incomes are high enough.
4.1 Symmetric Bayesian-Nash equilibrium without any reserve price

We now compute the symmetric Bayesian Nash equilibria for first-price, second-price and all-pay auctions without any reserve price.

4.1.1 First price auction

Let \( b^I (m_i) \), which is strictly increasing in \( m_i \) be the symmetric Bayesian Nash equilibrium in the first-price auction. Let bidders \( 2, 3, n \) choose \( b^I (m_2), b^I (m_3), \ldots b^I (m_n) \) and let bidder 1 choose a bid \( b_1 \). Note that \( b_1 \in [b^I (\alpha), b^I (\beta)] \). Since \( b^I (.) \) is strictly increasing there exists a \( z \in [\alpha, \beta] \) such that \( b_1 = b^I (z) \). Then the probability that 1 wins is

\[
\text{Prob.} \left( b_1 > \max \{b^I (m_2), b^I (m_3), \ldots b^I (m_n) \} \right)
\]

\[
= \text{Prob.} \left( b^I (z) > \max \{b^I (m_2), b^I (m_3), \ldots b^I (m_n) \} \right)
\]

\[
= \text{Prob.} \left( z > \max \{m_2, m_3, \ldots m_n \} \right)
\]

\[
= G(z) \text{ (see section 2.2 for the relevant notations)}
\]

Note that if 1 wins he pays \( b^I (z) \) to obtain good \( y \) and can spend the rest i.e. \( m_1 - b^I (z) \) on good \( x \) (whose price is unity). Conditional on winning, his payoff is \( q (m_1 - b^I (z)) + s \). If he does not win the auction he spends his entire income on good \( x \) and earns a payoff \( wm_1 \). Hence bidder 1’s expected payoff by bidding \( b^I (z) \) is

\[
\pi_1 (z, m_1) = G(z) \left[ q (m_1 - b^I (z)) + s \right] + (1 - G(z)) \cdot wm_1
\]

\[
= G(z) \left[ (q - w) m_1 + s - qb^I (z) \right] + wm_1.
\]

Now for \( b^I (m_1) \) to be the equilibrium bid chosen by bidder 1 we need that

\[
\frac{\partial \pi_1 (.)}{\partial z} = g(z) \left[ (q - w) m_1 + s - qb^I (z) \right] - qG(z) \frac{db^I (z)}{dz}
\]

\[
= 0 \text{ at } z = m_1.
\]

That is,

\[
g(m_1) \left[ (q - w) m_1 + s - qb^I (m_1) \right] - qG(m_1) \frac{db^I (m_1)}{dm_1} = 0 - - - - (1).
\]

Hence \( b^I (m_1) \) solves the following differential equation (2) and the boundary condition (2a).

\[
\frac{db^I}{dm_1} = \frac{g(m_1) \left[ (q - w) m_1 + s - qb^I \right]}{qG(m_1)} - - - - (2)
\]

\[
b^I (\alpha) = v(\alpha) = \frac{(q - w) \alpha + s}{q} - - - - (2a).
\]
To get the exact functional form of $b^I (m)$ we proceed as follows. Note that

$$
\frac{d}{dz} \left[ G(z) \left[ (q - w) z + s - qb^I(z) \right] \right]
= g(z) \left[ (q - w) m_1 + s - qb^I(z) \right] + G(z) \left[ (q - w) - q \frac{db^I(z)}{dz} \right]
= (q - w) G(z) \text{ (using (1)).}
$$

Since $G(\alpha) = 0$ we have

$$
G(z) \left[ (q - w) z + s - qb^I(z) \right] = \int_\alpha^z (q - w) G(t) \, dt
\Rightarrow b^I(z) = \frac{1}{q} \left[ (q - w) z + s - \frac{(q - w)}{G(z)} \int_\alpha^z G(t) \, dt \right].
$$

Noting that $G(.) = F^{n-1}(.)$ we get

$$
b^I(m) = \frac{1}{q} \left[ (q - w) m + s - \frac{(q - w)}{F^{n-1}(m)} \int_\alpha^m F^{n-1}(t) \, dt \right].
$$

4.1.2 Second-price auction

From proposition 1 we know that the following constitutes a symmetric Bayesian Nash equilibrium in the second price auction.

$$
b^H(m) = \frac{(q - w) m + s}{q}
$$

4.1.3 All-Pay auction

Let $b^{AP}(m_i)$, which is strictly increasing in $m_i$ be the symmetric Bayesian Nash equilibrium in the all-pay auction. Let bidders 2, 3..n choose $b^{AP}(m_2), b^{AP}(m_3), \ldots b^{AP}(m_n)$ and as before let bidder 1 choose a bid $b_1 = b^{AP}(z)$. Consequently, as before the probability that 1 wins is $G(z)$.

Note that in an all-pay auction bidder 1 pays $b^{AP}(z)$ irrespective of whether he wins or loses. If 1 wins he pays $b^{AP}(z)$ to obtain good y and can spend the rest i.e. $m_1 - b^{AP}(z)$ on good x (whose price is one). Conditional on winning, his payoff is $q \left( m_1 - b^{AP}(z) \right) + s$. If he loses he does not obtain good y but still pays $b^{AP}(z)$ and earns a payoff equal to $w \left( m_1 - b^{AP}(z) \right)$. Hence, bidder 1’s expected payoff when he bids $b^{AP}(z)$ is

$$
\pi_1(z, m_1) = G(z) \left[ q \left( m_1 - b^{AP}(z) \right) + s \right] + (1 - G(z)) \left[ w \left( m_1 - b^{AP}(z) \right) \right]
= G(z) \left[ (q - w) \left( m_1 - b^{AP}(z) \right) + s \right] + w \left( m_1 - b^{AP}(z) \right).
$$
Now for $b^{AP}(m_1)$ to be the equilibrium bid chosen by bidder 1 we need that
\[
\frac{\partial \pi_1(.)}{\partial z} = g(z) \left[ (q - w) (m_1 - b^{AP}(z)) + s \right] - (q - w) G(z) \frac{db^{AP}(z)}{dz} - w \frac{db^{AP}(z)}{dz} = 0 \text{ at } z = m_1.
\]
That is,
\[
g(m_1) \left[ (q - w) (m_1 - b^{AP}(m_1)) + s \right] - (q - w) G(m_1) \frac{db^{AP}(m_1)}{dm_1} - w \frac{db^{AP}(m_1)}{dm_1} = 0 \tag{3}
\]
Note that in a symmetric equilibrium $b^{AP}(\alpha) = 0$. The reason is that any bidder whose income is $\alpha$ wins with probability zero in a symmetric increasing equilibrium. Since it’s an all-pay auction he should bid zero and pay zero. Hence $b^{AP}(m_1)$ solves the following differential equation (4) and the boundary condition (4a).
\[
\frac{db^{AP}}{dm} = \frac{g(m_1) \left[ (q - w) (m_1 - b^{AP}) + s \right]}{G(m_1) q + (1 - G(m_1)) w} \tag{4}
\]
\[b^{AP}(\alpha) = 0 \tag{4a}.
\]
To get the exact functional form of $b^{AP}(m)$ we proceed as follows. Note that
\[
\frac{d}{dz} \left[ G(z) \left[ (q - w) (z - b^{AP}(z)) + s \right] + w \left( z - b^{AP}(z) \right) \right] = g(z) \left[ (q - w) (z - b^{AP}(z)) + s \right] + (q - w) G(z) \left( 1 - \frac{db^{AP}(z)}{dz} \right) + w \left( 1 - \frac{db^{AP}(z)}{dz} \right)
\]
\[= (q - w) G(z) + w \tag{using 3} \]
Since $G(\alpha) = 0$ and $b^{AP}(\alpha) = 0$ we get that
\[
G(z) \left[ (q - w) (z - b^{AP}(z)) + s \right] + w \left( z - b^{AP}(z) \right) = wz + \int_{\alpha}^{z} [(q - w) G(t) + w] dt.
\]
Noting that $G(.) = F^{n-1}(.)$ from above we get
\[
b^{AP}(m) = \frac{F^{n-1}(m) [(q - w) m + s] - (q - w) \int_{\alpha}^{m} F^{n-1}(t) dt}{F^{n-1}(m) q + (1 - F^{n-1}(m)) w}.
\]
\textbf{Remark 1} The derivations of $b^{f}(.)$ and $b^{AP}(.)$ are only heuristic because both (1) and (3) are necessary conditions. We have not formally established that if the other $(n - 1)$ bidders follow $b^{f}(.)$ or $b^{AP}(.)$ then it is indeed optimal for a bidder with income $m$ to bid $b^{f}(m)$ or $b^{AP}(m)$. We demonstrate this in the appendix after the proof of proposition 9.
4.1.4 Expected revenues without any reserve price

The expected revenue in the three auctions (without any reserve price) are as follows.

\[
\begin{align*}
\text{First-price} & : R^I = \int_{\alpha}^{\beta} b^I(m) f_1(m) \, dm \\
\text{Second-price} & : R^{II} = \int_{\alpha}^{\beta} b^{II}(m) f_2(m) \, dm \\
\text{All-pay} & : R^{AP} = n \int_{\alpha}^{\beta} b^{AP}(m) f(m) \, dm
\end{align*}
\]

We now provide our next main result.

**Proposition 2** When bidders are risk neutral and \( \alpha \geq k = \frac{s}{w} \), then \( R^{AP} > R^I = R^{II} \).

**Remark 2** With risk neutral bidders, the expected revenue is same in first-price and second-price auctions for high enough incomes. Surprisingly however, this does not extend to all-pay auctions. This stands in stark contrast to the benchmark model where the expected revenue is same for all three auctions analysed here. We illustrate this in an example.

**Example 5** Let \( u(x, y) = x(1+y) + y \)

Here \( q = 2, w = 1 \) and \( s = 1 \). Let \( \alpha = 1, \beta = 2 \) and let \( F(.) \) be uniform on \([1, 2]\). Here \( k = 1 \) and \( u(m_i, 0) \geq u(0, 1) \) for all \( m_i \in [1, 2] \).

Routine computation shows that \( c(m_i) = \frac{m_i - 1}{2} \) and hence \( v_i = m_i - c(m_i) = \frac{m_i + 1}{2} \). We also have the following. \( b^I(m) = \frac{m+3}{4} \), \( b^{II}(m) = \frac{m+1}{2} \) and \( b^{AP}(m) = \frac{1}{2}m - \frac{3}{2m} + 1 \). The expected revenues are as follows.

\[
\begin{align*}
R^I &= \frac{7}{6} = 1.1667 \\
R^{II} &= \frac{7}{6} = 1.1667 \\
R^{AP} &= \frac{7}{2} - 3\ln 2 = 1.4206.
\end{align*}
\]

Clearly all-pay auction fetches more revenue and the revenue equivalence breaks down even with risk neutral bidders.
4.2 Introducing reserve prices in the risk neutral case

Let the auctioneer impose a reserve price $r$. This means that bids below $r$ are not acceptable. As before we let let $\alpha \geq k = \frac{w}{q}$ (all incomes are high enough). This implies at for all $m_i \in [\alpha, \beta]$, $v(m_i) = \frac{q-w}{q} m_i + s$ . It may be noted that without any reserve price we had $b^I(\alpha) = v(\alpha)$, $b^{II}(\alpha) = v(\alpha)$ and $\beta^{AP}(\alpha) = 0$. Consequently, any reserve price, $r$, such that $0 < r < v(\alpha)$ will have no impact on bidding equilibrium in first-price and second price auctions. However, it will affect the equilibrium in all-pay auctions. Also note that, if $r > v(\beta)$ then in any auction the optimum action for any bidder is to choose $\{No\}$ (that is, not participate in the auction for good $y$, and spend the entire income on good $x$). Hence, we restrict our attention to $r \in [0, v(\beta)]$. We give below the symmetric Bayesian Nash equilibrium for all the three auctions when the seller imposes a positive reserve price $r$. The computation of such equilibria are straightforward.

4.2.1 First-Price auction

If $r \in [0, v(\alpha))$ then for all $m \in [\alpha, \beta]$

$$b^I(m) = \frac{1}{q} \left[ (q-w) m + s - \frac{q-w}{F^{n-1}(m)} \int_\alpha^m F^{n-1}(t) \, dt \right] \text{ for all } m \in [\alpha, \beta].$$

If $r \in [v(\alpha), v(\beta)]$ then

$$b^I(m) = \begin{cases} 
\{No\} & \text{if } m \in \left[ \alpha, \frac{qr-s}{q-w} \right] \\
\frac{1}{q} \left[ (q-w) m + s - \frac{q-w}{F^{n-1}(m)} \int_{\alpha}^{m} F^{n-1}(t) \, dt \right] & \text{if } m \in \left[ \frac{qr-s}{q-w}, \beta \right]. 
\end{cases}$$

4.2.2 Second-Price auction

If $r \in [0, v(\alpha))$ then for all $m \in [\alpha, \beta]$

$$b^{II}(m) = \frac{1}{q} \left[ (q-w) m + s \right] \text{ for all } m \in [\alpha, \beta].$$

If $r \in [v(\alpha), \ v(\beta)]$ then

$$b^{II}(m) = \begin{cases} 
\{No\} & \text{if } m \in \left[ \alpha, \frac{qr-s}{q-w} \right] \\
\frac{(q-w)m+s}{q} & \text{if } m \in \left[ \frac{qr-s}{q-w}, \beta \right]. 
\end{cases}$$
4.2.3 All-pay auction

Let \( r \in [0, \infty) \) and let in a symmetric equilibrium \( \mu \) be the level of income (which depends on \( r \)) such that a bidder with income \( \mu \) is indifferent between bidding and not bidding. If the bidder with income \( \mu \) chooses not to bid (that is, he selects \( \{No\} \)) then he spends his entire income on good \( x \) and gets utility \( w\mu \). If he bids, he must bid the lowest admissible amount which is \( r \). If he bids \( r \) then the probability that he wins in a symmetric equilibrium is \( F^{n-1}(\mu) \), the probability that all other incomes are less than \( \mu \). His expected payoff is then

\[
F^{n-1}(\mu) [q(\mu - r) + s] + (1 - F^{n-1}(\mu)) w(\mu - r)
\]

The above must be equal to \( w\mu \), since he is indifferent between bidding and not bidding. That is, we have

\[
F^{n-1}(\mu) [q(\mu - r) + s] + (1 - F^{n-1}(\mu)) w(\mu - r) = w\mu
\]

This implies

\[
F^{n-1}(\mu) [(q - w)(\mu - r) + s] = wr.
\]

We can now easily compute that if \( r \in [0, v(\beta)] \) then

\[
b^{AP}(m) = \begin{cases} 
\{No\} & \text{if } m \in [\alpha, \mu) \\
\frac{F^{n-1}(m)[(q-w)m+s]-(q-w) \int_{m}^{\mu} F^{n-1}(t) dt}{F^{n-1}(m)[q(1-F^{n-1}(m))]} & \text{if } m \in [\mu, \beta]
\end{cases}
\]

where \( \mu \) is s.t. \( F^{n-1}(\mu) [(q - w)(\mu - r) + s] = wr \).

4.2.4 Expected revenues with a reserve price

We now deduce the expected revenue going to the seller for each of the three auctions with a reserve price \( r \). In a first price auction the expected payment of a bidder with income \( m \geq \frac{qr-s}{q-w} \) is

\[
P^f(m, r) = \text{(Prob. win)} \times b^f(m)
= F^{n-1}(m) \times b^f(m).
= F^{n-1}(m) \frac{(q-w)m+s}{q} - \frac{q-w}{q} \int_{\frac{qr-s}{q-w}}^{m} F^{n-1}(t) dt
\]
Similarly in a second price auction the expected payment of a bidder with income \( m = \frac{qr-s}{q-w} \) is 
\[ rF^{n-1} \left( \frac{qr-s}{q-w} \right) \] 
and the expected payment of a bidder with income \( m > \frac{qr-s}{q-w} \) is 
\[ P^{II} (m, r) = rF^{n-1} \left( \frac{qr-s}{q-w} \right) + \frac{\int_{\frac{qr-s}{q-w}}^{m} \left( q - w \right) z + s \, dF^{n-1} (z)}{q} \]

In an all-pay auction the expected payment of a bidder with income \( m \geq \mu \) is just \( b^{AP} (m) \). That is, 
\[ P^{AP} (m, r) = \frac{F^{n-1} (m) [(q - w) m + s] - (q - w) \int_{\mu}^{m} F^{n-1} (t) \, dt}{F^{n-1} (m) q + (1 - F^{n-1} (m)) w} \]

The expected revenues are as follows.

First-Price Auction:
\[ R^{I} (r) = n \int_{\frac{qr-s}{q-w}}^{\beta} P^{I} (m, r) f (m) \, dm \]

Second-Price Auction:
\[ R^{II} (r) = n \int_{\frac{qr-s}{q-w}}^{\beta} P^{II} (m, r) f (m) \, dm \]

All-Pay Auction:
\[ R^{AP} (r) = n \int_{\mu}^{\beta} P^{AP} (m, r) f (m) \, dm \]
where \( \mu \) is s.t. \( F^{n-1} (\mu) [(q - w) (\mu - r) + s] = wr \).

We now provide our next set of main results.

**Proposition 3** \( R^{I} (r) = R^{II} (r) \) for all \( r \in [0, v (\beta)] \).

**Remark 3** This result is identical to the one in the symmetric benchmark model, where for any reserve price the expected revenue is same in first price and second price auctions.

We next compute the optimal reserve price in each of these auctions. Let
\[ r^{*} (FPA) = \arg \max_{r \geq 0} R^{I} (r) \]
\[ r^{*} (SPA) = \arg \max_{r \geq 0} R^{II} (r) \]
and
\[ r^{*} (APA) = \arg \max_{r \geq 0} R^{AP} (r) \]

Note that proposition 3 clearly shows that the optimum reserve price will be the same in first-price and second price auction. That is, \( r^{*} (FPA) = r^{*} (SPA) \).
Proposition 4 If the hazard rate \( \frac{f(\lambda)}{1-F(\lambda)} \) is non-decreasing and if \( f(\alpha) \geq \frac{q-w}{(q-w)\alpha+s} \) then

\[
 r^* (FPA) = r^* (SPA) = r^* (APA) = 0.
\]

Remark 4 It may be noted that in the benchmark model the revenue maximising reserve price is always positive\(^6\). From proposition 2 we know that \( R^{AP}(0) > R^I(0) = R^{II}(0) \). Consequently, from the seller’s viewpoint the all pay auction with zero reserve price is the best. This stands in contrast to the standard symmetric model of auctions. Whether or not this is the optimal mechanism in this context remains an open question.

Remark 5 It may also be noted that non-decreasing hazard rate is a very standard assumption in the benchmark auction model. Note that higher is \( \alpha \) (the minimum possible income) the lower is \( \frac{q-w}{(q-w)\alpha+s} \) and the inequality \( f(\alpha) \geq \frac{q-w}{(q-w)\alpha+s} \) is more likely to be satisfied. As such, we can intuitively say that if all incomes are very high then the optimal reserve price will always be zero.

5 Equilibrium in first-price and all-pay auctions with low enough incomes

We now turn our attention to first price auction and all-pay auction with low enough incomes. (i.e. \( \beta < k \)). This means \( m_i < k \) for all \( i \) and \( v(m_i) = m_i \) for all \( i \). Also \( u(0,1) - u(m_i,0) > 0 \) for all \( m_i \in [\alpha, \beta] \). Let us denote the marginal utility of \( x \) by the following.

\[
 MU_x(x,y) = \frac{\partial u(x,y)}{\partial x} = u'(x,y) \quad \text{(this is an abuse of notation)}.
\]

We now provide our next two results. The first one deals with first-price auctions.

Proposition 5 If \( \beta < k \) and if the reverse hazard rate \( \frac{f(\lambda)}{1-F(\lambda)} \) is non-increasing and if for all \( x \in [0,\beta] \)

\[
u'(x,1) \leq (n-1) f(\beta) [u(0,1) - u(\beta,0)]
\]

then \( b^I(m_i) = m_i \) is a symmetric Bayesian Nash equilibrium in a first price auction.

\(^6\)In the benchmark model if \( F_v(\cdot) \) and \( f_v(\cdot) \) are the distribution and density functions of valuations, the revenue maximising \( r^* \) solves \( r^* = \frac{1-F_v(r^*)}{f_v(r^*)} = 0. \)
**Remark 6** Note that for the above result to hold true $\beta < k$ is essential as it ensures that $u(0, 1) - u(\beta, 0) > 0$. If $\beta = k$ then $u(0, 1) - u(\beta, 0) = 0$ and this would mean the requirement

$$u'(x, 1) \leq (n - 1) f(\beta)[u(0, 1) - u(\beta, 0)]$$

cannot be satisfied (as marginal utility is strictly positive). Also note that if $\frac{\partial^2 u(x, y)}{\partial x^2} \leq 0$ (the bidders are risk averse) then we need

$$u'(0, 1) \leq (n - 1) f(\beta)[u(0, 1) - u(\beta, 0)].$$

**Proposition 6** If $\alpha = 0$ and $\beta < k$ and if $\forall m_i \in [0, \beta]$

$$F^{n-1}(m_i) u(0, 1) \geq u(m_i, 0)$$

and

$$F^{n-1}(m_i) u(0, 1) \geq F^{n-1}(m_i - \varepsilon) u(\varepsilon, 1) + (1 - F^{n-1}(m_i - \varepsilon)) u(\varepsilon, 0), \ \forall \varepsilon \in (0, m_i).$$

then $b^{AP}(m_i) = m_i$ is a symmetric Bayesian Nash equilibrium in an all-pay auction.

**Remark 7** Note that in any increasing symmetric equilibrium of the all-pay auction $b^{AP}(\alpha) = 0$. The reason is that any bidder whose income is $\alpha$ wins with probability zero in a symmetric increasing equilibrium. Since it’s an all-pay auction he should bid zero and pay zero. Hence, we need $\alpha = 0$ if $b^{AP}(m_i) = m_i$ is to be a symmetric Bayesian Nash equilibrium in an all-pay auction. If all the conditions of both propositions 5 and 6 hold then all bidders bidding their valuations is a symmetric Bayesian Nash equilibrium for all the three auctions.

The expected revenues in this case are as follows.

First-price : $R' = \int_0^\beta zf_1(z) \, dz,$

Second-price : $R'' = \int_0^\beta zf_2(z) \, dz$

All-Pay : $R^{AP} = n \int_0^\beta zf(z) \, dz$

We now provide the revenue ranking result for the case when incomes are low enough.
Proposition 7 If all the conditions of propositions 5 and 6 hold then \( R^{AP} > R^{I} > R^{II} \).

Remark 8 When bidders’ incomes are low enough revenue equivalence fails to hold. In fact, like the case of risk neutral bidders with high enough incomes all-pay auction fetches the highest expected revenue. We have not analysed the effect of reserve price on the equilibria with low enough incomes. Such an exercise and the computation of optimal reserve price should be an interesting area of research.

We now provide an example to illustrate proposition 7.

Example 4 Let \( u(x, y) = x + 3y, n = 2 \) and let \( M_i \) be uniformly distributed over \([0, 1]\). That is, \( F(m) = m \). Note that here \( k = 3 \). Here \( R^{I} = \frac{2}{3}, R^{II} = \frac{1}{3} \) and \( R^{AP} = 1 \).

6 Extension: Equilibrium with risk averse bidders and high enough incomes

We now proceed to analyse existence of equilibrium when bidders are risk averse with high enough incomes.

Let \( \alpha \geq k \) (that is, incomes are high enough). Then \( u(0, 1) < u(m_i, 0) \) for all \( m_i \in (\alpha, \beta] \). This implies \( v(m_i) = m_i - c(m_i) \) for all \( m_i \in [\alpha, \beta] \).

Note that if \( u'(s, 1) \geq u'(t, 0) \) for all \( (s, t) \in [0, \beta - v(\beta)] \times [\alpha, \beta] \) then \( v'(\cdot) \geq 0 \). The next two propositions spell out the results for first-price and all-pay auctions without any reserve price.

Proposition 8 If \( \frac{\partial^2 u(x, 1)}{\partial x^2} \leq 0 \) for all \( x \in [0, \beta] \) and \( u'(s, 1) \geq u'(t, 0) \) for all \( (s, t) \in [0, \beta - v(\beta)] \times [\alpha, \beta] \), then there exists a symmetric Bayesian-Nash equilibrium in a first-price auction which solves the following differential equation with the boundary condition.

\[
\begin{align*}
\frac{db'^I}{dm} &= (n - 1) \frac{f(m_i)}{F(m_i)} \left[ \frac{u(m_i - b', 1) - u(m_i, 0)}{u'(m_i - b', 1)} \right] \\
b'(\alpha) &= v(\alpha)
\end{align*}
\]
Proposition 9 If $\frac{\partial^2 u(x,1)}{\partial x^2} \leq 0$ for all $x \in [0, \beta]$ and $u' (s, 1) \geq u' (t, 0)$ for all $(s, t) \in [0, \beta] \times [0, \beta]$, then there exists a symmetric Bayesian-Nash equilibrium in an all-pay auction which solves the following differential equation with the boundary condition.

$$\frac{db^{AP}}{dm} = \frac{(n - 1) F^{n-2} (m_i) f (m_i) [u (m_i - b^{AP}, 1) - u (m_i - b^{AP}, 0)]}{F^{n-1} (m_i) u' (m_i - b^{AP}, 1) + (1 - F^{n-1} (m_i)) u' (m_i - b^{AP}, 0)}$$

$b^{AP} (\alpha) = 0$.

Remark 9 While we have provided the sufficient conditions for existence of symmetric Bayesian-Nash equilibrium in first-price and second-price auction, a revenue ranking result and the effects of reserve prices in this context are still open research problems.

7 Conclusion

In this paper we had a relook at the basic auction model. Unlike the benchmark model we consider a more general class of utility functions and assume that all individuals have the same utility functions but have different incomes. We treat incomes as types. Each individual’s income is private information and this implies that valuations (that are functions of income) are also private information. We have shown that in two goods world where one good is available in a store at a fixed price and the other good is sold in an auction, many results of the benchmark model do not hold. While Vickrey’s (1961) result on second price auction is very robust, revenue equivalence breaks down even with risk-neutral bidders.

We propose the following for future research.

1. In this paper for first-price and all-pay auctions we have dealt with either high enough incomes (i.e. $\alpha \geq k$) or low enough incomes (i.e. $\beta < k$). We have not analysed the case when $\alpha < k < \beta$. Derivation of equilibrium in first price auctions and all-pay auctions, effects of reserve price and revenue ranking results when $\alpha < k < \beta$ should be an interesting (and challenging) future course of research.

2. It would also be interesting to compute the expected revenue maximising optimal mechanisms in our model.

We believe there is ample scope for further research into this area.
References


Appendix

Proof of Proposition 1 We know that

\[
v(m_i) = \begin{cases} 
  m_i & \text{if } m_i \leq k \\
  m_i - c(m_i) & \text{if } m_i > k 
\end{cases}
\]

where \( c(m_i) \) is such that \( u(c(m_i), 1) = u(m_i, 0) \).

Let’s consider bidder 1 whose income is \( m_1 \). Let \( z \) be the maximum of the others’ bids. Let player 1 bid \( b_1 \). There are two possible cases.

Case 1 \( m_1 \leq k \). This means \( v(m_1) = m_1 \). For this case there are three possible subcases.

Subcase (i) \( z > m_1 \). If 1 chooses \( b_1 = m_1 \) he gets a payoff equal to \( u(m_1, 0) \) (as he lososes in the auction and spends his entire income on good \( x \)). Any \( b_1 < z \) will give him the same payoff. If he bid \( b_1 \geq z \) he wins with a positive probability. Note that if \( b_1 > z \) he wins with certainty and if \( b_1 = z \) there is a tie and his probability of winning is same as the probability of winning the tie. Since \( z > m_1 \) bidder 1 cannot pay for the object after winning it. This means he will default and he has to pay a positive penalty. Hence for this subcase \( b_1 = m_1 \) is the best.

Subcase (ii) \( z < m_1 \). Choosing a bid \( b_1 = m_1 \) gives player 1 a payoff equal to \( u(m_1 - z, 1) \) (as he wins with certainty and pays \( z \) for one unit of good \( y \)). If he chooses \( b_1 \in (z, m_1) \) he gets the same payoff. Any bid strictly greater than \( m_1 \) will also fetch him the same payoff. If he chooses \( b_1 < z \) he lososes the auction and gets \( u(m_1, 0) \). Since \( m_1 \leq k \) we have \( u(m_1, 0) \leq u(0, 1) < u(m_1 - z, 1) \). If he chooses \( b_1 = z \) then there is a tie and let \( \rho \) be the probability that he wins the tie. His payoff will be \( \rho u(m_1 - z, 1) + (1 - \rho) u(m_1, 0) \leq u(m_1 - z, 1) \) for all \( \rho \in [0, 1] \). Hence his best bid is \( b_1 = m_1 \).

Subcase (iii) \( z = m_1 \). If he chooses \( b_1 = m_1 = z \) he gets \( \rho u(0, 1) + (1 - \rho) u(m_1, 0) \). If he chooses \( b_1 < m_1 = z \) he gets \( u(m_1, 0) \). Since \( m_1 \leq k \) we have \( u(m_1, 0) \leq u(0, 1) \) and hence we have \( \rho u(0, 1) + (1 - \rho) u(m_1, 0) \geq u(m_1, 0) \) for all \( \rho \in [0, 1] \). If \( b_1 > z = m_1 \) bidder 1 wins with certainty but cannot pay for the object after winning it. This means he will default and he has to pay a positive penalty. Hence for this subcase also \( b_1 = m_1 \) is the best.

Case 2 \( m_1 > k \). This means \( v(m_1) = m_1 - c(m_1) \). For this case also there are three possible subcases.

Subcase (i) \( z > v(m_1) \). If \( b_1 = v(m_1) \) bidder 1 looses in the auction and gets \( u(m_1, 0) \). Any bid \( b_1 < v(m_1) \) gives him the same payoff. Note that we can rule out any bid strictly
greater than \( m_1 \). Hence if \( z \geq m_1 \) then bidder 1’s best option is to choose a bid \( v(m_1) \). If \( z < m_1 \) then if \( b_1 \in (z, m_1] \) will fetch 1 a payoff equal to \( u(m_1 - z, 1) \). Since \( z > v(m_1) \) we have \( u(m_1 - z, 1) < u(m_1 - v(m_1), 1) = u(c(m_1), 1) = u(m_1, 0) \). Hence \( b_1 = v(m_1) \) is better than \( b_1 \in (z, m_1] \). If \( b_1 = z \) bidder 1 wins with probability \( \rho \) and gets

\[
\rho u(m_1 - z, 1) + (1 - \rho) u(m_1, 0)
\]

\[
\leq \rho u(m_1 - v(m_1), 1) + (1 - \rho) u(m_1, 0)
\]

\[
= \rho u(c(m_1), 1) + (1 - \rho) u(m_1, 0)
\]

\[
= u(m_1, 0) \text{ since } u(c(m_1), 1) = u(m_1, 0).
\]

Hence for this subcase \( b_1 = v(m_1) \) is the best.

Subcase (ii) \( z < v(m_1) \). Here if \( b_1 = v(m_1) \) bidder 1 wins in the auction and gets \( u(m_1 - z, 1) \). If he chooses \( b_1 \in (z, v(m_1)) \) he gets the same payoff. If he chooses \( b_1 < z \) he looses in the auction and gets \( u(m_1, 0) \). Note that since \( z < v(m_1) \) we have \( u(m_1 - z, 1) > u(m_1 - v(m_1), 1) = u(c(m_1), 1) = u(m_1, 0) \). Hence \( b_1 = v(m_1) \) is strictly better than \( b_1 < z \). If \( b_1 = z \) bidder 1 wins with probability \( \rho \) and gets \( \rho u(m_1 - z, 1) + (1 - \rho) u(m_1, 0) \leq u(m_1 - z, 1) \) for all \( \rho \in [0, 1] \). Hence for this subcase \( b_1 = v(m_1) \) is the best.

Subcase (iii) \( z = v(m_1) \). Here if \( b_1 = v(m_1) \) bidder 1 wins in the auction with probability \( \rho \) and gets

\[
\rho u(m_1 - z, 1) + (1 - \rho) u(m_1, 0)
\]

\[
= \rho u(m_1 - v(m_1), 1) + (1 - \rho) u(m_1, 0)
\]

\[
= \rho u(c(m_1), 1) + (1 - \rho) u(m_1, 0)
\]

\[
= u(m_1, 0) \text{ since } u(c(m_1), 1) = u(m_1, 0).
\]

If he chooses a bid \( b \in (z, m_1) = (v(m_1), m_1) \) then he wins with certainty and gets \( u(m_1 - z, 1) = u(m_1 - v(m_1), 1) = u(c(m_1), 1) = u(m_1, 0) \). If he chooses \( b_1 < z \) he gets \( u(m_1, 0) \). Hence \( b_1 = v(m_1) \) is the best bid for bidder 1 in this subcase.

24
Proof of Proposition 2  First note that from section 4.11 and 4.14 we have

\[
R^I = \int_{\alpha}^\beta b^I (m) f_1 (m) \, dm
\]

\[
= \frac{1}{q} \int_{\alpha}^\beta \left[ (q-w) m + s - \frac{(q-w)}{F^{n-1} (m)} \int_{\alpha}^m F^{n-1} (t) \, dt \right] f_1 (m) \, dm
\]

\[
= \frac{1}{q} \left[ \int_{\alpha}^\beta [(q-w) m + s] f_1 (m) \, dm - (q-w) \int_{\alpha}^\beta \frac{1}{F^{n-1} (m)} \left( \int_{\alpha}^m F^{n-1} (t) \, dt \right) f_1 (m) \, dm \right] - - (5)
\]

Also note that

\[
\int_{\alpha}^\beta [(q-w) m + s] f_1 (m) \, dm
\]

\[
= (q-w) \int_{\alpha}^\beta m f_1 (m) \, dm + s - - - - (6).
\]

We also have

\[
(q-w) \int_{\alpha}^\beta \frac{1}{F^{n-1} (m)} \left( \int_{\alpha}^m F^{n-1} (t) \, dt \right) f_1 (m) \, dm
\]

\[
= (q-w) \int_{\alpha}^\beta \frac{1}{F^{n-1} (m)} \left( \int_{\alpha}^m F^{n-1} (t) \, dt \right) n F^{n-1} (m) f (m) \, dm \text{ (see section 2.2)}
\]

\[
= n (q-w) \int_{\alpha}^\beta \left( \int_{\alpha}^m F^{n-1} (t) \, dt \right) f (m) \, dm
\]

\[
= n (q-w) \int_{\alpha}^\beta \left( \int_{t}^\beta f (m) \, dm \right) F^{n-1} (t) \, dt \text{ (by interchanging the order of integration)}
\]

\[
= n (q-w) \int_{\alpha}^\beta (1 - F (t)) F^{n-1} (t) \, dt
\]

\[
= (q-w) \int_{\alpha}^\beta n (1 - F (m)) F^{n-1} (m) \, dm - - - - (7)
\]

Using (6) and (7) in (5) we get

\[
R^I = \frac{1}{q} \left[ (q-w) \int_{\alpha}^\beta m f_1 (m) \, dm + s - (q-w) \int_{\alpha}^\beta n (1 - F (m)) F^{n-1} (m) \, dm \right] - - - - (8)
\]

This means

\[
R^I = \frac{q-w}{q} \left[ \int_{\alpha}^\beta m f_1 (m) \, dm - \int_{\alpha}^\beta n (1 - F (m)) F^{n-1} (m) \, dm \right] + \frac{s}{q} - - - - (9).
\]

Note that

\[
R^{II} = \int_{\alpha}^\beta b^{II} (m) f_2 (m) \, dm
\]

\[
= \int_{\alpha}^\beta \frac{(q-w) m + s}{q} f_2 (m) \, dm
\]

\[
= \frac{(q-w)}{q} \int_{\alpha}^\beta m f_2 (m) \, dm + \frac{s}{q} - - - - (10).
\]
Hence

\[
R^I - R^{II} = \frac{(q - w)}{q} \left[ \int_\alpha^\beta m f_1 (m) \, dm - \int_\alpha^\beta n (1 - F (m)) F^{n-1} (m) \, dm - \int_\alpha^\beta m f_2 (m) \, dm \right]
\]

\[
= \frac{(q - w)}{q} \left[ \int_\alpha^\beta m F_1 (m) \, dm - \int_\alpha^\beta n (1 - F (m)) F^{n-1} (m) \, dm - \int_\alpha^\beta m F_2 (m) \, dm \right]
\]

\[
= \frac{(q - w)}{q} \left[ [m F_1 (m)]_\alpha^\beta F_1 (m) \, dm - [m F_2 (m)]_\alpha^\beta n (1 - F (m)) F^{n-1} (m) \, dm \right]
\]

\[
- [m F_1 (m)]_\alpha^\beta + [m F_2 (m)]_\alpha^\beta F_2 (m) \, dm
\]

\[
= \frac{(q - w)}{q} \left[ \beta - \int_\alpha^\beta F^n (m) \, dm - \int_\alpha^\beta n (1 - F (m)) F^{n-1} (m) \, dm \right]
\]

\[
- \beta + \int_\alpha^\beta [F^n (m) + n F^{n-1} (m) (1 - F (m))] \, dm
\]

\[
= 0.
\]

Therefore

\[
R^I = R^{II}.
\]

Note that

\[
R^{AP} = n \int_\alpha^\beta b^{AP} (m) f (m) \, dm - - - - (11).
\]

Now

\[
b^{AP} (m) = \frac{F^{n-1} (m) [(q - w) m + s] - (q - w) \int_\alpha^m F^{n-1} (t) \, dt}{F^{n-1} (m) q + (1 - F^{n-1} (m)) w}.
\]

Note that \(q > w\) and \(F^{n-1} (m) \in (0, 1)\) for all \(m \in (\alpha, \beta)\). This means

\[
\text{for all } m \in (\alpha, \beta) \quad q > F^{n-1} (m) q + (1 - F^{n-1} (m)) w.
\]

Hence

\[
b^{AP} (m) > \frac{1}{q} \left[ F^{n-1} (m) [(q - w) m + s] - (q - w) \int_\alpha^m F^{n-1} (t) \, dt \right].
\]

From (11) we get

\[
R^{AP} > \frac{1}{q} \int_\alpha^\beta \left[ F^{n-1} (m) [(q - w) m + s] - (q - w) \int_\alpha^m F^{n-1} (t) \, dt \right] n f (m) \, dm
\]

\[
= \frac{1}{q} \left[ \int_\alpha^\beta [(q - w) m + s] n F^{n-1} (m) f (m) \, dm - n (q - w) \int_\alpha^\beta \left( \int_\alpha^m F^{n-1} (t) \, dt \right) f (m) \, dm \right]
\]

\[
= \frac{1}{q} \left[ \int_\alpha^\beta [(q - w) m + s] f_1 (m) \, dm - n (q - w) \int_\alpha^\beta \left( \int_\alpha^m F^{n-1} (t) \, dt \right) f (m) \, dm \right] - (12).
\]
It may be noted that

\[
\int_{\alpha}^{\beta} \left( \int_{\alpha}^{m} F^{n-1} (t) \, dt \right) f (m) \, dm \\
= \int_{\alpha}^{\beta} \left( \int_{t}^{\beta} f (m) \, dm \right) F^{n-1} (t) \, dt \text{ (changing the order of differentiation).} \\
= \int_{\alpha}^{\beta} (1 - F (t)) F^{n-1} (t) \, dt -- -- (13)
\]

Using (13) in (12) we get

\[
R^{AP} > \frac{1}{q} \left[ \int_{\alpha}^{\beta} [(q - w) m + s] f_1 (m) \, dm - n (q - w) \int_{\alpha}^{\beta} (1 - F (m)) F^{n-1} (m) \, dm \right] \\
= R^I \text{ (using 6 and 8).}
\]

Therefore \( R^{AP} > R^I = R^{II} \).  \( \blacksquare \)

**Proof of Proposition 3** When \( r \in [0, v (\alpha)] \) then it makes no difference to the equilibrium in both first-price auction and second-price auction. That is, \( r \in [0, v (\alpha)] \) is equivalent to \( r = 0 \).

With \( r = 0 \) we know that \( R^I = R^{II} \) (from proposition 2).

We also know that for all \( m \in \left[ \frac{q r - s}{q - w}, \beta \right] \)

\[
P^I (m, r) = F^{n-1} (m) \left( \frac{q - w}{q} m + s \right) \frac{q - w}{q} \int_{\frac{q r - s}{q - w}}^{m} F^{n-1} (t) \, dt -- -- (14)
\]

Similarly for all \( m \in \left[ \frac{q r - s}{q - w}, \beta \right] \)

\[
P^{II} (m, r) = r F^{n-1} \left( \frac{q r - s}{q - w} \right) + \int_{\frac{q r - s}{q - w}}^{m} \frac{(q - w) z + s}{q} dF^{n-1} (z) -- -- (15)
\]

Now since \( F^{n-1} (.) = G (.) \) we get

\[
P^{II} (m, r) = r G \left( \frac{q r - s}{q - w} \right) + \int_{\frac{q r - s}{q - w}}^{m} \frac{(q - w) z + s}{q} dG (z) \\
= r G \left( \frac{q r - s}{q - w} \right) + \left[ \frac{(q - w) z + s}{q} G (z) \right]_{\frac{q r - s}{q - w}}^{m} - \int_{\frac{q r - s}{q - w}}^{m} G (z) d \left( \frac{(q - w) z + s}{q} \right) \\
= r G \left( \frac{q r - s}{q - w} \right) + \frac{(q - w) m + s}{q} G (m) - r G \left( \frac{q r - s}{q - w} \right) - \frac{(q - w)}{q} \int_{\frac{q r - s}{q - w}}^{m} G (z) \, dz \\
= \frac{(q - w) m + s}{q} G (m) - \frac{(q - w)}{q} \int_{\frac{q r - s}{q - w}}^{m} G (z) \, dz \\
= \frac{(q - w) m + s}{q} F^{n-1} (m) - \frac{(q - w)}{q} \int_{\frac{q r - s}{q - w}}^{m} F^{n-1} (z) \, dz \text{ (since } F^{n-1} (.) = G (.) \text{)} \\
= P^I (m, r) \text{ (from 14).} \]
We also know that for \( r \in [v(\alpha), v(\beta)] \)

\[
R^I(r) = n \int_{\frac{q-r}{q-w}}^{\beta} P^I(m, r) f(m) \, dm
\]

\[
R^{II}(r) = n \int_{\frac{q-r}{q-w}}^{\beta} P^{II}(m, r) f(m) \, dm
\]

Using (14) and from discussion at the beginning of the proof we get that \( R^I(r) = R^{II}(r) \) for all \( r \in [0, v(\beta)] \).

**Proof of Proposition 4** First note that for any \( r \in [0, v(\alpha)] \), \( R^I(r) \) is same as \( R^I(0) \). And so

\[
r \in [0, v(\alpha)] \text{ we have } \frac{d}{dr} R^I(r) = 0.
\]

For \( r \in (v(\beta), \infty) \) we have \( R^I(r) = 0 \) (since no bidder will make any bid for such a \( r \)).

For \( r \in [v(\alpha), v(\beta)] \) we have

\[
\frac{d}{dr} R^I(r) = \frac{d}{dr} \left[ n \int_{\frac{q-r}{q-w}}^{\beta} P^I(m, r) f(m) \, dm \right]
\]

\[
= n \left[ -\frac{q}{q-w} P^I \left( \frac{q-r-s}{q-w}, r \right) f \left( \frac{q-r-s}{q-w} \right) + \int_{\frac{q-r-s}{q-w}}^{\beta} \frac{\partial}{\partial r} \left( P^I(m, r) f(m) \right) \, dm \right] - - - (15).\]

Now

\[
P^I \left( \frac{q-r-s}{q-w}, r \right) = F^{n-1} \left( \frac{q-r-s}{q-w}, \frac{q-w}{q} \left( \frac{q-r-s}{q-w} \right) + s \right)
\]

\[
= r F^{n-1} \left( \frac{q-r-s}{q-w} \right) - - - (16)
\]

And

\[
\frac{\partial}{\partial r} \left( P^I(m, r) \right) = \frac{\partial}{\partial r} \left[ F^{n-1}(m) \frac{(q-w)m + s - q-w}{q} \int_{\frac{q-r-s}{q-w}}^{m} F^{n-1}(t) \, dt \right]
\]

\[
= \frac{q-w}{q} \times \frac{q-w}{q} F^{n-1} \left( \frac{q-r-s}{q-w} \right)
\]

\[
= F^{n-1} \left( \frac{q-r-s}{q-w} \right) - - - (17).
\]
Using (16) and (17) in (15) we get that

\[ \frac{d}{dr} R^I (r) = n \left[ - \frac{q}{q-w} r F^{n-1} \left( \frac{q r - s}{q-w} \right) f \left( \frac{q r - s}{q-w} \right) + \int_{\frac{q r - s}{q-w}}^{\beta} F^{n-1} \left( \frac{q r - s}{q-w} \right) f (m) \, dm \right] \]

\[ = n \left[ - \frac{q}{q-w} r F^{n-1} \left( \frac{q r - s}{q-w} \right) f \left( \frac{q r - s}{q-w} \right) + F^{n-1} \left( \frac{q r - s}{q-w} \right) \int_{\frac{q r - s}{q-w}}^{\beta} f (m) \, dm \right] \]

\[ = n \left[ - \frac{q}{q-w} r F^{n-1} \left( \frac{q r - s}{q-w} \right) f \left( \frac{q r - s}{q-w} \right) + F^{n-1} \left( \frac{q r - s}{q-w} \right) \left( 1 - F \left( \frac{q r - s}{q-w} \right) \right) \right] \]

\[ = n F^{n-1} \left( \frac{q r - s}{q-w} \right) \left( 1 - F \left( \frac{q r - s}{q-w} \right) \right) \left[ - \frac{q r}{q-w} f \left( \frac{q r - s}{q-w} \right) \left( \frac{q r}{q-w} - 1 \right) + 1 \right] \] - - (18).

Since \( \frac{f(z)}{1-F(z)} \) is non-decreasing in \( z \) and \( f (\alpha) \geq \frac{q-w}{(q-w) \alpha + s} \), we have

\[ \frac{f \left( \frac{q r - s}{q-w} \right)}{1 - F \left( \frac{q r - s}{q-w} \right)} \geq \frac{f (\alpha)}{1 - F (\alpha)} = f (\alpha) \geq \frac{q-w}{(q-w) \alpha + s} \] - - - (19).

Now

\[ r \in [v (\alpha), v (\beta)] \Rightarrow r \geq \frac{(q-w) \alpha + s}{q} \] - - - (20).

Using (19) and (20) we get

\[ \left( \frac{q r}{q-w} \right) f \left( \frac{q r - s}{q-w} \right) \left( \frac{q r}{q-w} - 1 \right) \geq \left[ q \left( \frac{(q-w) \alpha + s}{q} \right) \right] \left( \frac{q-w}{(q-w) \alpha + s} \right) = 1 \] - - - (21).

From (21) and (18) we get that \( \frac{d}{dr} R^I (r) \leq 0 \) for all \( r \in [v (\alpha), v (\beta)] \). Note that \( R^I (r) = R^I (0) \) for all \( r \in [0, v (\alpha)] \). This implies that \( r^* (FPA) = 0 \). Since \( R^I (r) = R^I (r) \) for all \( r \in [0, v (\beta)] \) (from proposition 3) we also get that \( r^* (SPA) = 0 \).

To compute \( r^* (APA) \) note the following. We know that if \( r \in [0, v (\beta)] \) then

\[ b^{AP} (m) = \begin{cases} \{ No \} & \text{if } m \in [\alpha, \mu) \, F^{n-1} (m) \mid (q-w) m + s \mid (q-w) \int_{\mu}^{m} F^{n-1} (t) \, dt \, F^{n-1} (m) q + (1 - F^{n-1} (m)) w \, \text{if } m \in [\mu, \beta] \end{cases} \]

where \( \mu \) is s.t. \( F^{n-1} (\mu) [(q-w) (\mu-r) + s] = wr. \)

In an all-pay auction the expected payment of a bidder with income \( m \geq \mu \) is just \( b^{AP} (m) \). That is,

\[ p^{AP} (m, r) = \frac{F^{n-1} (m) [(q-w) m + s] - (q-w) \int_{\mu}^{m} F^{n-1} (t) \, dt}{F^{n-1} (m) q + (1 - F^{n-1} (m)) w}. \]
The expected revenues for any \( r \in [0, v(\beta)] \) is as follows.

\[
R^{AP}(r) = n \int_{\mu}^{\beta} P^{AP}(m, r) f(m) \, dm
\]

Hence

\[
\frac{d}{dr} R^{AP}(r) = n \left[ -\frac{d\mu}{dr} P^{AP}(\mu, r) f(\mu) + \int_{\mu}^{\beta} \frac{\partial}{\partial r} \left( P^{AP}(m, r) \right) f(m) \, dm \right] - - - (23)
\]

Note that

\[
P^{AP}(\mu, r) f(\mu) = \frac{F^{-1}(\mu) \left[ (q - w) \mu + s \right] f(\mu)}{F^{-1}(\mu) q + (1 - F^{-1}(\mu)) w} - - - (24)
\]

Also

\[
\frac{\partial}{\partial r} \left( P^{AP}(m, r) \right) = \frac{(q - w) \frac{d\mu}{dr} F^{-1}(\mu)}{F^{-1}(m) q + (1 - F^{-1}(m)) w} - - - (25)
\]

Note that since since \( q > w \) and \( \mu \leq m \) we get

\[
F^{-1}(\mu) q + (1 - F^{-1}(\mu)) w \leq F^{-1}(m) q + (1 - F^{-1}(m)) w.
\]

Using the above in (25) we have

\[
\frac{\partial}{\partial r} \left( P^{AP}(m, r) \right) \leq \frac{(q - w) \frac{d\mu}{dr} F^{-1}(\mu)}{F^{-1}(\mu) q + (1 - F^{-1}(\mu)) w} - - - (26)
\]

From (24) and (26) and from (23) we get

\[
\frac{d}{dr} R^{AP}(r) \leq n \left[ - \frac{d\mu}{dr} \frac{F^{-1}(\mu) \left[ (q - w) \mu + s \right] f(\mu)}{F^{-1}(\mu) q + (1 - F^{-1}(\mu)) w} + \int_{\mu}^{\beta} \frac{(q - w) \frac{d\mu}{dr} F^{-1}(\mu) f(m) \, dm}{F^{-1}(\mu) q + (1 - F^{-1}(\mu)) w} \right]
\]

\[
= \frac{n F^{-1}(\mu) \frac{d\mu}{dr}}{F^{-1}(\mu) q + (1 - F^{-1}(\mu)) w} \left[ - [(q - w) \mu + s] f(\mu) + (q - w) \int_{\mu}^{\beta} f(m) \, dm \right]
\]

\[
= \frac{n F^{-1}(\mu) \frac{d\mu}{dr}}{F^{-1}(\mu) q + (1 - F^{-1}(\mu)) w} \left[ - [(q - w) \mu + s] f(\mu) + (q - w) (1 - F(\mu)) \right]
\]

\[
= \frac{n F^{-1}(\mu) \frac{d\mu}{dr}}{F^{-1}(\mu) q + (1 - F^{-1}(\mu)) w} \left[ - [(q - w) \mu + s] f(\mu) \right] + (q - w) \] - - - (27).
\]

Since \( \mu \) is s.t. \( F^{-1}(\mu) \left[ (q - w) (\mu - r) + s \right] = wr \) we get that \( \mu \geq \alpha \) and \( \frac{d\mu}{dr} \geq 0 \). Also it is given that \( \frac{f(z)}{1 - F(z)} \) is non-decreasing in \( z \) and \( f(\alpha) \geq \frac{q - w}{(q - w)\alpha + s} \). So we get

\[
\frac{[(q - w) \mu + s] f(\mu)}{1 - F(\mu)} \geq \frac{[(q - w) \alpha + s] f(\alpha)}{1 - F(\alpha)} = [(q - w) \alpha + s] f(\alpha) \geq (q - w) \] - - - (28).
\]

Hence from (27) and (28) we get \( \frac{d}{dr} R^{AP}(r) \leq 0 \). This implies \( r^*(APA) = 0 \).
Proof of Proposition 5  Let bidders 2, 3, ..., n bid $m_2, m_3, ..., m_n$. Bidder 1’s income is $m_1$ and he chooses a bid equal to $b_1$. Then bidder 1’s expected payoff is

$$
\pi_1 (b_1, m_1) = G (b_1) u (m_1 - b_1, 1) + (1 - G (b_1)) u (m_1, 0).
$$

Then

$$
\frac{\partial \pi_1 (.)}{\partial b_1} = g (b_1) [u (m_1 - b_1, 1) - u (m_1, 0)] - G (b_1) u' (m_1 - b_1, 1).
$$

First consider the case where $m_1 = \alpha$. Since $b_1 \leq m_1 = \alpha$ and for all $b_1 \in [0, \alpha]$ we have $G (b_1) = 0$ and $u (m_1 - b_1, 1) - u (m_1, 0) > 0$, therefore the best possible $b_1$ is $\alpha$. That is, $b^I (\alpha) = \alpha$.

Now consider the case where $m_1 > \alpha$. If bidder 1 chooses $b_1 = \alpha$, then $\frac{\partial \pi_1 (.)}{\partial b_1} > 0$. Hence for this case $b_1 > \alpha$. We can write $\frac{\partial \pi_1 (.)}{\partial b_1}$ as follows (note that $G (b_1) > 0$ for $b_1 > \alpha$).

$$
\frac{\partial \pi_1 (.)}{\partial b_1} = G (b_1) \left[ \frac{g (b_1)}{G (b_1)} [u (m_1 - b_1, 1) - u (m_1, 0)] - u' (m_1 - b_1, 1) \right]
$$

$$
= G (b_1) \left[ (n - 1) \frac{f (b_1)}{F (b_1)} [u (m_1 - b_1, 1) - u (m_1, 0)] - u' (m_1 - b_1, 1) \right].
$$

The above follows because

$$
\frac{g (b_1)}{G (b_1)} = (n - 1) \frac{f (b_1)}{F (b_1)}.
$$

Since $b_1 \leq m_1$ and since the reverse hazard rate $\frac{f (b_1)}{F (b_1)}$ is non-increasing we have

$$
(n - 1) \frac{f (b_1)}{F (b_1)} \leq (n - 1) \frac{f (m_1)}{F (m_1)}.
$$

Since $u (.)$ is strictly increasing in $x$ we have

$$
(n - 1) \frac{f (b_1)}{F (b_1)} [u (m_1 - b_1, 1) - u (m_1, 0)]
$$

$$
> (n - 1) \frac{f (m_1)}{F (m_1)} [u (0, 1) - u (m_1, 0)]
$$

$$
\geq (n - 1) \frac{f (\beta)}{F (\beta)} [u (0, 1) - u (\beta, 0)]
$$

$$
= (n - 1) f (\beta) [u (0, 1) - u (\beta, 0)]
$$

$$
\geq u' (x, 1) \text{ (as per the hypothesis of the proposition)}.
$$

Hence when $m_1 > \alpha$, we have $\frac{\partial \pi_1 (.)}{\partial b_1} > 0$ for all $b_1 \leq m_1$. Therefore the best $b_1$ is $m_1$. That is, $b^I (m_1) = m_1$. }
Proof of Proposition 6  Let bidders 2, 3,...,n bid \(m_2, m_3,\ldots, m_n\). Bidder 1’s income is \(m_1\) and he chooses a bid equal to \(b_1\). Then bidder 1’s expected payoff is

\[
\pi_1 (b_1, m_1) = G (b_1) u (m_1 - b_1, 1) + (1 - G (b_1)) u (m_1 - b_1, 0)
\]

\[
= F^{n-1} (b_1) u (m_1 - b_1, 1) + (1 - F^{n-1} (b_1)) u (m_1 - b_1, 0).
\]

If bidder 1 chooses \(b_1 = m_1\) his expected payoff is \(F^{n-1} (b_1) u (0, 1)\). If he chooses not to bid then he gets \(u (m_1, 0)\). If he chooses a bid strictly less than \(m_1\) he chooses \(b_1 = m_1 - \varepsilon\) where \(\varepsilon \in (0, m_1)\). Then his expected payoff is

\[
F^{n-1} (m_1 - \varepsilon) u (\varepsilon, 1) + (1 - F^{n-1} (m_1 - \varepsilon)) u (\varepsilon, 0).
\]

Since \(\forall m_i \in [0, \beta]\) we have (by the hypotheses of the proposition)

\[
F^{n-1} (m_i) u (0, 1) \geq u (m_i, 0)
\]

and \(F^{n-1} (m_i) u (0, 1) \geq F^{n-1} (m_i - \varepsilon) u (\varepsilon, 1) + (1 - F^{n-1} (m_i - \varepsilon)) u (\varepsilon, 0), \forall \varepsilon \in (0, m_i)\)

choosing \(b_1 = m_1\) is optimal.■

Proof of Proposition 7  The expected revenues in this case are as follows.

First-price:

\[
R^I = \int_0^\beta z f_1 (z) \, dz = \int_0^\beta zdF_1 (z)
\]

Second-price:

\[
R^{II} = \int_0^\beta z f_2 (z) \, dz = \int_0^\beta zdF_2 (z)
\]

All-Pay:

\[
R^{AP} = n \int_0^\beta z f (z) \, dz
\]

First note that \(F_1 (z) = F^n (z)\) and \(F_2 (z) = F^n (z) + n F^{n-1} (z) (1 - F (z))\). Since \(F_1 (z) < F_2 (z)\) for all \(z \in (\alpha, \beta)\) we have \(R^I > R^{II}\).

Also note that

\[
R^I = \int_0^\beta z f_1 (z) \, dz = n \int_0^\beta z F^{n-1} (z) f (z) \, dz
\]

\[
< n \int_0^\beta z f (z) \, dz \text{ (since } F^{n-1} (z) < 1 \text{ for all } z \in [0, \beta])
\]

\[
= R^{AP}.
\]

Hence we have \(R^{AP} > R^I > R^{II}\).■
Proof of Proposition 8  Let $b^l (m_i)$, which is strictly increasing in $m_i$ be the symmetric Bayesian Nash equilibrium in the first-price auction. Let bidders 2,3..n choose $b^l (m_2), b^l (m_3), ... b^l (m_n)$ and let bidder 1 choose a bid $b_1$. Note that $b_1 \in [b^l (\alpha), b^l (\beta)]$. Since $b^l (.)$ is strictly increasing there exists a $z \in [\alpha, \beta]$ such that $b_1 = b^l (z)$. Then the probability that 1 wins is $G(z)$. Bidder 1’s expected payoff by bidding $b^l (z)$ is

$$
\pi_1 (z, m_1) = G(z) u (m_1 - b^l (z), 1) + (1 - G(z)) u (m_1, 0)
= G(z) [u (m_1 - b^l (z), 1) - u (m_1, 0)] + u (m_1, 0).
$$

Note that

$$
\frac{\partial \pi_1 (.)}{\partial z} = g(z) \frac{G(z) u (m_1 - b^l (z), 1) - u (m_1, 0) - G(z) u' (m_1 - b^l (z), 1) \frac{db^l (m_1)}{dm}}{dz} = 0 \text{ at } z = m_1.
$$

That is

$$
g(m_1) [u (m_1 - b^l (m_1), 1) - u (m_1, 0)] - G(m_1) u' (m_1 - b^l (m_1), 1) \frac{db^l (m_1)}{dm} = 0
$$

Note that $G(m_1) = F^{n-1}(m_1)$ and $g(m_1) = (n-1) F^{n-2}(m_1) f(m_1)$. Hence $b^l (m_1)$ solves the following differential equation (29) and the boundary condition (29a).

$$
\frac{db^l}{dm} = (n-1) \frac{f(m) [u (m - b^l (m), 1) - u (m, 0)]}{F(m) u' (m - b^l (m), 1)} = v(m) \text{ (29)}
$$

$$
b^l (\alpha) = v(\alpha). \text{ (29a)}
$$

Note that (29) and (29a) are only necessary conditions. To show the sufficient condition we define

$$
J(z) = \pi_1 (m_1, m_1) - \pi_1 (m_1, z)
$$

If all bidders 2,3..n follow strategy $b^l (.)$ then $\pi_1 (m_1, m_1)$ is the expected payoff to player 1 when he bids $b_1 = b^l (m_1)$ and $\pi_1 (m_1, z)$ is the expected payoff to player 1 when he bids $b^l (z)$. If $J(z) \geq 0$ for all $z \neq m_1$ then 1’s best bid is $b^l (m_1)$ and $b^l (.)$ will indeed constitute a symmetric Bayesian-Nash equilibrium. Let us also denote the following.

$$
u(x, 1) = H(x) \text{ and } u(x, 0) = h(x)
$$

Then

$$
J(z) = G(m_1) H (m_1 - b^l (m_1)) + (1 - G(m_1)) h (m_1)
- G(z) H (m_1 - b_{33}^l (z)) - (1 - G(z)) h (m_1).
$$
Note that

\[ J' (z) = g (z) H (m_1 - b^I (z)) + G (z) H' (m_1 - b^I (z)) \frac{db^I (z)}{dz} + h (m_1) g (z) \]

Substituting for \( \frac{db^I (z)}{dz} \) from (29) and noting that \((n - 1) \frac{f (x)}{f (x)} = \frac{g (x)}{G (x)} \) we get

\[
J' (z) = -g (z) H (m_1 - b^I (z)) + G (z) H' (m_1 - b^I (z)) \frac{g (z) \left( H (z - b (z)) - h (z) \right)}{G (z)} + h (m_1) g (z)
\]

\[
= g (z) \left[ \frac{H' (m_1 - b^I (z))}{H' (z - b (z))} \left[ H (z - b^I (z)) - h (z) \right] - \left[ H (m_1 - b^I (z)) - h (m_1) \right] \right] - (32).
\]

If \( z > m_1 \) then \( H' (m_1 - b^I (z)) \geq H' (z - b (z)) \) (since \( \frac{\partial^2 u(x,1)}{\partial x^2} \leq 0 \)). This means

\[
z > m_1 \implies J' (z) \geq \left[ H (z - b^I (z)) - h (z) \right] - \left[ H (m_1 - b^I (z)) - h (m_1) \right]
\]

\[
= \left[ H (z - b^I (z)) - H (m_1 - b^I (z)) \right] + [h (z) - h (m_1)] - - - - - (33)
\]

Using the mean-value theorem we get when \( z > m_1 \) there exists \( s \in (m_1 - b^I (z), z - b^I (z)) \) and \( t \in (m_1, z) \) such that

\[
[H (z - b^I (z)) - H (m_1 - b^I (z))] - [h (m_1) - h (z)]
\]

\[
= (z - b^I (z) - m_1 + b^I (z)) H' (s) - (z - m_1) h' (t)
\]

\[
= (z - m_1) \left[ H' (s) - h' (t) \right] - - - - - (34).
\]

From the hypothesis of the proposition we know that \( H' (s) - h' (t) \geq 0 \) and hence from (33) and (34) we have

\[
z > m_1 \implies J' (z) \geq 0 - - - - - (35).
\]

By a similar logic we can show that

\[
z < m_1 \implies J' (z) \leq 0 - - - - - (36).
\]

(35) and (36) imply that the \( J (z) \) reaches a minimum when \( z = m_1 \) and since \( J (m_1) = 0 \) we get that \( J (z) \geq 0 \) for all \( z \neq m_1 \). This means that if bidders 2, 3...n choose bids \( b^I (m_2), b^I (m_3) ... b^I (m_n) \) it is best for bidder 1 to choose bid \( b^I (m_1) \).
Proof of Proposition 9  Let \( b_{AP}(m_i) \), which is strictly increasing in \( m_i \) be the symmetric Bayesian Nash equilibrium in the all-pay auction.

Let bidders 2, 3..n choose \( b_{AP}(m_2), b_{AP}(m_3), ..., b_{AP}(m_n) \) and let bidder 1 choose a bid \( b_1 \). Note that \( b_1 \in [b_{AP}(\alpha), b_{AP}(\beta)] \). Since \( b_{AP}(.) \) is strictly increasing there exists a \( z \in [\alpha, \beta] \) such that \( b_1 = b_{AP}(z) \). Then the probability that 1 wins is \( G(z) \). Bidder 1’s expected payoff by bidding \( b_1^l(z) \) is

\[
\pi_1(z,m_1) = G(z)u(m_1 - b_{AP}(z), 1) + (1 - G(z))u(m_1 - b_{AP}(z), 0)
\]

\[
= G(z)[u(m_1 - b_1^l(z), 1) - u(m_1 - b_{AP}(z), 0)] + u(m_1 - b_{AP}(z), 0)
\]

We now closely follow the proof of proposition 8 and can show that \( b_{AP}(m) \) solves the following differential equation (37) and the boundary condition (37a).

\[
\frac{db_{AP}}{dm} = \frac{(n - 1) F^{n-2}(m_i) f(m_i)[u(m_i - b_{AP}, 1) - u(m_i - b_{AP}, 0)]}{F^{n-1}(m_i) u'(m_i - b_{AP}, 1) + (1 - F^{n-1}(m_i)) u'(m_i - b_{AP}, 0)} - - - (37)
\]

\[
b_{AP}(\alpha) = 0, - - - - - (37a)
\]

We now borrow notations from the proof of proposition 8 and define

\[
J(z) = \pi_1(m_1,m_1) - \pi_1(m_1,z)
\]

Here we have

\[
J(z) = G(m_1)H(m_1 - b_{AP}(m_1)) + (1 - G(m_1))h(m_1 - b_{AP}(m_1))
\]

\[
- G(z)H(m_1 - b_{AP}(z)) - (1 - G(z))h((m_1 - b_{AP}(z))) - - - (38).
\]

\[
J'(z) = -g(z)H(m_1 - b_{AP}(z)) + G(z)H'(m_1 - b_{AP}(z)) \frac{db_{AP}(z)}{dz} + h'(m_1 - b_{AP}(z)) \frac{db_{AP}(z)}{dz}
\]

\[
+ g(z)h(m_1 - b_{AP}(z)) - G(z)h'(m_1 - b_{AP}(z)) \frac{db_{AP}(z)}{dz}
\]

\[
= g(z)[-H(m_1 - b_{AP}(z)) + h(m_1 - b_{AP}(z))]
\]

\[
+ \frac{db_{AP}(z)}{dz}[G(z)H'(m_1 - b_{AP}(z)) + h'(m_1 - b_{AP}(z)) - G(z)h'(m_1 - b_{AP}(z))] - (39)
\]

From (37) we know that

\[
\frac{db_{AP}(z)}{dz} = \frac{g(z)[H(z - b_{AP}(z)) - h(z - b_{AP}(z))]}{G(z)H'(z - b_{AP}(z)) + (1 - G(z))h'(z - b_{AP}(z))} - - - (40)
\]

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Substituting for \(\frac{db^{AP}(z)}{dx}\) from (40) into (39) and cancelling terms we get

\[
J'(z) = g(z) \left[\left[-H\left(m_1 - b^{AP}(z)\right) + h\left(m_1 - b^{AP}(z)\right)\right] + \left[H\left(z - b^{AP}(z)\right) - h\left(z - b^{AP}(z)\right)\right]\right] - - - (41).
\]

Using the mean-value theorem we get when \(z > m_1\) there exists \(s_1 \in (m_1 - b^I(z), z - b^I(z))\) and \(t_1 \in (m_1 - b^I(z), z - b^I(z))\) such that

\[
\left[H\left(z - b^{AP}(z)\right) - H\left(m_1 - b^{AP}(z)\right)\right] - \left[h\left(z - b^{AP}(z)\right) - h\left(m_1 - b^{AP}(z)\right)\right] = (z - b^I(z) - m_1 + b^I(z)) H'(s_1) - (z - m_1) h'(t_1)
\]

\[
= (z - m_1) [H'(s_1) - h'(t_1)] - - - (42).
\]

From the hypothesis of the proposition we know that \(H'(s_1) - h'(t_1) \geq 0\) and hence from (41) and (42) we have

\[
z > m_1 \implies J'(z) \geq 0 - - - (43).
\]

By a similar logic we can show that

\[
z < m_1 \implies J'(z) \leq 0 - - - (44).
\]

(43) and (44) imply that the \(J(z)\) reaches a minimum when \(z = m_1\) and since \(J(m_1) = 0\) we get that \(J(z) \geq 0\) for all \(z \neq m_1\). This means that if bidders 2, 3...n choose bids \(b^{AP}(m_2), b^{AP}(m_3) ... b^{AP}(m_n)\) it is best for bidder 1 to choose bid \(b^{AP}(m_1)\). ■

**Sufficient conditions for the risk neutral case with high enough incomes**  For the risk neutral case we have \(\frac{\partial^2 u(x,1)}{\partial x^2} = 0\). Also note that for all \(x\) we have \(u'(x,1) = q > w = u'(x,0)\). This means all the conditions of propositions 8 and 9 hold true for the risk neutral case as well. Hence a similar logic will show that if all bidders 2, 3...n choose either bids \(b^{AP}(m_2), b^{AP}(m_3) ... b^{AP}(m_n)\) or bids \(b^I(m_2), b^I(m_3), ... b^I(m_n)\) then it is best for bidder 1 to choose either \(b^{AP}(m_1)\) or \(b^I(m_1)\) as the case may be.

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