Abstract

I identify assumptions under which policies that maximize expected surplus are Pareto Optimal—even when expected consumer surplus does not even locally represent preferences over price-income lotteries. Besides the oft-made partial equilibrium assumptions that only one price varies, and that income changes do not affect demand, the two other assumptions are that every consumer’s indirect utility function is supermodular in price and income; and policies order prices by a single-crossing property. Supermodularity holds precisely if relative risk aversion exceeds the income elasticity of demand, a mild condition. The single-crossing property appears strong, but holds in many applications that use expected surplus.

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1 Introduction

It is well-understood that consumer’s surplus cannot represent a consumer’s preferences over all positive price-income pairs (Chipman and Moore (1976)). Yet, in a model without uncertainty, surplus-maximizing policies often just equate prices and marginal costs, which is (first-best) Pareto Optimal. For example, per-unit taxes should be set equal to zero; and a monopolist should be subsidized until price equals marginal cost. This fact surely helps explain the widespread use of total surplus to evaluate polices in applied microeconomics, allowing economists to avoid specifying details of consumer preferences and endowments.

It is easy to see why policies which maximize surplus are often Pareto Optimal in models with no uncertainty and complete markets: for small price and income changes, the change in a consumer’s surplus is proportional to the change in that consumer’s utility. If equilibrium prices and incomes are constrained to move along a particular path—for example, they only move in response to changes in taxes—then consumer’s surplus locally represents a consumer’s preferences over prices and income on the path.
With uncertainty and incomplete markets, it seems unlikely that policies which maximize expected surplus will be Pareto Optimal: even if incomes and the prices of all but one good remain fixed, the change in expected consumer’s surplus is not proportional to change in expected utility unless the marginal utility of income is constant in the price which changes (Rogerson (1980) and Turnovsky, Shalit, and Schmitz (1980)). And polices which maximize expected total surplus often do not have the obvious intuitive appeal of “price = marginal cost” in the certainty case. For example, Deneckere, Marvel, and Peck (1997) find that allowing a monopolist manufacturer to impose a price floor for retailers can raise expected total and expected consumers’ surplus. And Baron and Myerson (1982) find that the optimal regulatory policy for a monopolist with private cost information can involve prices which are higher than the highest possible price of an unregulated monopoly. Curiously, defenses of total surplus in Industrial Organization texts only refer to the certainty case–e.g, Tirole (1988), Introduction, and Vives (1999), Chapter 3–although most applications in Industrial Organization involve uncertainty.

Although expected consumer’s surplus does not in general even locally represent a consumer’s preferences over policies which affect the random price of a good, I find that policies which maximize it are Pareto Optimal under two assumptions–in addition to the standard if strong assumptions that the policy only affects one relative price and that there are no income effects on demand (Theorem 1). One assumption is that each consumer’s indirect utility function is supermodular in income and the price of the good, equivalently, the marginal utility of money is increasing in the price of the good; or each consumer’s relative risk aversion over money lotteries is at least as large as the income elasticity of demand for the good. Since income elasticities must on average equal 1, this assumption is empirically plausible. The other assumption is that any expected surplus maximizing policy generates a random price which is less variable (in a sense to be made precise) than any other available policy. This assumption seems to be a strong one, but it holds in many applications, and I identify environments in which it does: roughly, it is more likely to hold if uncertainty is about consumers rather than producers. I confirm that under the plausible restriction of supermodular indirect utilities, minimum resale price maintenance remains Pareto Optimal in the model of Deneckere, Marvel, and Peck (1997), but the policy identified by Baron and Myerson (1982) is not–indeed all consumers and the firm can prefer no regulation at all to the Baron-Myerson policy.

To show that expected-surplus-maximizing policies are Pareto Optimal, I establish a slightly stronger result on an individual consumer’s preferences: if a consumer strictly prefers policy A to B, then that consumer’s expected surplus is higher under A than under B.

1In a related paper, Schlee (2008) exploits this fact to show that the percentage error from using expected consumer’s surplus to approximate the willingness to pay for a price change is unbounded, in contrast to Willig (1976)’s famous approximation result in the certainty case (Hayashi (2008) has an updated version of this result). Here I am not concerned with how well expected surplus measures the welfare loss, but whether maximizing it identifies optimal policies.
2 Preliminaries

The economy has $I$ consumers, $J$ firms, and $L$ goods ($L \geq 2$). Good $L$ is *numeraire*. All consumers face the same prices, which they take as given. The policy affects the price of good 1, and possibly also the prices of other goods and the income of some consumers. Relabel the goods so that the prices of goods $n + 1$ through $L$ are not affected by the policy ($1 \leq n \leq L - 1$). Important special cases are $n = 1$ (the policy only affects one relative price) and $n = L - 1$ (it affects them all). The price vector is $P \in \mathbb{R}^L_{++}$. Consumer $i$’s income, $m_i$, equals the value of its endowment plus its share of profits plus any transfers:

$$m_i = P \cdot e_i + \sum_j \theta_{ij} \Pi_j + t_i,$$

(1)

where $(0 \neq) e_i \in \mathbb{R}^L_+$ is consumer $i$’s endowment of goods, $\theta_{ij} \geq 0$ is consumer $i$’s ownership share of firm $j$, $\Pi_j \geq 0$ is firm $j$’s profit, and $t_i$ is any transfer that consumer $i$ receives. Each consumer’s preferences over goods is represented by twice continuously differentiable, strongly monotone, and strongly quasiconcave utility function. Consumer $i$’s indirect utility function is $V^i(P, m_i)$, and demand vector is $D^i(P, m_i)$. Subscripts on $V^i$ denote partial derivatives: $V^i_\ell$ is the derivative of $V^i$ with respect to the price of good $\ell$, and $V^i_m$ is the derivative of $V^i$ with respect to money. I use lowercase variables for good 1: $p$ is the price of good 1 and $d^i$ is consumer $i$’s demand for good 1.

**The Problem**

For small changes in prices and income (using Roy’s Identify and supressing the consumer superscript $i$)

$$dV = V_P \cdot dP + V_m dm = V_m [-D \cdot dP + dm] = dS \times V_m,$$

or succinctly

$$dV = dS \times V_m.$$

(2)

Equation (2) formalizes the familiar fact already mentioned: the change in consumer’s surplus is proportional to the change in utility, so that the change in consumer’s surplus *locally* represents a consumer’s preferences over policies: $dV > 0$ if and only if $dS > 0$. Equation (2) also illustrates one clear problem with expected consumer’s surplus: $E[dS] > 0$ does not in general imply that $E[dV] > 0$ unless the marginal utility of money is constant in any variables that are random, as pointed out by Rogerson (1980) and Turnovsky, Shalit, and Schmitz (1980). The main goal is to find conditions on preferences and the set of feasible policies to ensure that any policy which maximizes expected surplus is Pareto Optimal.

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2By *strongly* monotone and *strongly* quasiconcave I mean in a differential sense: at every point in the domain of the function, all the first derivatives of the function are *positive*; and the Hessian matrix of the function is negative *definite* in the subspace of vectors orthogonal to the gradient vector of the function.
3 Certainty: Complete Markets

Before presenting the main results under uncertainty, I begin by reviewing how well total surplus finds optimal policies in the absence of uncertainty. For the certainty case the model similar to Radner (1993). I index policies by a real number $\alpha$; in this section I assume $\alpha$ lies in an interval $A = [\alpha, \bar{\alpha}]$, where $\bar{\alpha} > \alpha$. Consumers are price-takers, but I do not explicitly model producers or define equilibrium other than insisting that the aggregate production plan equals aggregate demand (so the results are consistent with many assumptions about firm behavior). I do assume that for each policy $\alpha \in A$, there is a unique equilibrium price vector $\Phi(\alpha)$ and distribution of income $(m_1(\alpha), ..., m_I(\alpha))$. In addition I assume (in this section) that equilibrium prices and the income distribution are continuously differentiable on $A$. Since consumers face the same prices, the allocation of the aggregate production to consumers is Pareto Optimal, so the only issue is optimality of the aggregate production plan.

The change in consumer $i$’s surplus between policy $\bar{\alpha}$ and $\alpha$ is the path integral

$$S_i(\alpha) = -\int_{[\alpha, \bar{\alpha}]} D^i(\Phi(\theta), m_i(\theta)) \cdot d\Phi(\theta) + m_i(\alpha) - m_i(0),$$

and total surplus is $S(\alpha) = \sum S_i(\alpha)$. Consumer $i$’s preferences over policies in $A$ are represented by the function $\alpha \mapsto V^i(\Phi(\alpha), m_i(\alpha))$.

Since I only consider a single path of prices and income—the one generated by the policy and the equilibrium condition—I bypass the usual difficulty that the change in consumers’ surplus between two price-income pairs depends on the path chosen. And since I only consider price-income pairs along this path, I don’t need to recover preferences over all price-income pairs to determine an optimal policy.

I start with a classic tax-subsidy example for which a surplus-maximizing policy is Pareto Optimal, which I will use as an illustration throughout.

Example 1 (A tax-subsidy problem.) Suppose that markets are competitive and that the economy’s technology is Leontief. In any competitive equilibrium with positive consumption of all goods, profit is zero and producer prices of all goods are determined solely by the technology. Let $\bar{p} > 0$ be the producer price of good 1. Let $\alpha \in [-1, 1]$ be the amount of a per-unit tax on good 1, and assume that this tax is the only distortion. Suppose that the net revenue raised from each consumer is rebated to that consumer: for each consumer $i$, the transfer is $t_i(\alpha) = \alpha d^i(p + \alpha, m_i + t_i(\alpha))$. To ensure that the rebate is well-defined assume that $|d^i_m| < 1$ globally. In addition assume that $d_p^i + d_m^i d_m^i < 0$ (the substitution effect on good 1 is negative, not just nonpositive). Differentiate $i$’s surplus with respect to $\alpha$ and rearrange to find that

$$S_i'(\alpha) = \alpha d_p^i + d_m^i d_m^i,$$

which is zero at $\alpha = 0$, positive when $\alpha < 0$ and negative when $\alpha > 0$. So surplus for each consumer is maximized at a zero tax, which is Pareto Optimal. Note that $S''_i(0) = d_1^i + d_2^i d_2^i < 0$ for every consumer $i$, so $S''(0) < 0$ and total surplus is strongly quasiconcave in the tax $\alpha$.

3 That is, each firm has a constant returns technology, there is no joint production, and there is only one nonproduced input. See for example Mas-cocell, Whinston, and Green (1995), Proposition 5.AA.2.
A function \( f \) on \( A \) is \textit{locally Pareto Consistent} if for every \( \hat{\alpha} \in [\underline{\alpha}, \bar{\alpha}] \) with \( f'(\hat{\alpha}) \neq 0 \), there is a neighborhood \( N \) of \( \hat{\alpha} \) such that \( f(\cdot) \) represents \textit{some} consumer \( i \)'s preferences over the policy \( \alpha \) on \( N \cap A \). Local Pareto Consistency is a weak requirement (but Expected Total Surplus violates it).

Apply equation (2) to find
\[
\frac{d}{d\alpha} V^i(\Phi(\alpha), m_i(\alpha)) = \frac{S'_i(\alpha)}{V'_m(\Phi(\alpha), m_i(\alpha))},
\] (4)
so the change in surplus from a small policy change is
\[
S'(\alpha) = \sum \frac{d}{d\alpha} V^i(\Phi(\alpha), m_i(\alpha)).
\] (5)
The next result follows immediately from (5): if \( S'(\hat{\alpha}) > 0 \), then some consumer’s preferences are increasing in the policy \( \alpha \) in a neighborhood of \( \hat{\alpha} \).

**Proposition 1** Total surplus is locally Pareto consistent.

Indeed, a stronger result is true: if \( S'(\hat{\alpha}) > 0 \), then a small increase in \( \alpha \) from \( \hat{\alpha} \) is a \textit{potential Pareto improvement} (that is, after the policy increase, the new supply can be allocated so that every consumer is better off with the higher policy). Let \( y(\alpha) \in \mathbb{R}^L \) be an equilibrium production plan, and suppose that it is unique and differentiable; denote its derivative by \( y'(\alpha) \). With the differentiability assumption on the equilibrium production plan, the economy fits the framework of Radner (1993). His main result is if \( \Phi(\hat{\alpha}) \cdot y'(\hat{\alpha}) > 0 \)—the value of the production change is positive at the starting equilibrium prices—then a small increase in \( \alpha \) from \( \hat{\alpha} \) is a potential Pareto improvement whenever \( S'(\hat{\alpha}) > 0 \). A closely related extension is that the change in total surplus is (locally) proportional to the \textit{coefficient of resource utilization} (Debreu (1954)), a measure of the welfare loss from a policy expressed as a percentage \( (\rho) \) of the economy’s endowment: Debreu (1954) shows that a first-order approximation of \( \rho \) is given by \( d\rho = \sum dV^i \times (P \cdot e)^{-1} \)–see his equation (5) in Section 3—which by equation (5) is proportional to the change in total surplus, \( dS \).

A policy \( \alpha' \in A \) is \textit{Pareto Optimal} if there is no other policy in \( A \) that every consumer weakly prefers to \( \alpha' \) and some consumer strictly prefers to \( \alpha' \). A policy \( \alpha' \in A \) is \textit{locally Pareto Optimal} if there is a neighborhood \( N \) of \( \alpha' \) such that \( \alpha' \) is Pareto Optimal on \( N \cap A \).

**Proposition 2** If \( \alpha^* \) maximizes surplus on \( A = [\underline{\alpha}, \bar{\alpha}] \), then it is locally Pareto Optimal under any one of the following conditions.

(a) \( S'_i(\alpha^*) \neq 0 \) for at least one consumer \( i \).

(b) \( S'(\alpha^*) \neq 0 \) (implying that \( \alpha^* \) is either \( \underline{\alpha} \) or \( \bar{\alpha} \));

(c) \( S''(\alpha^*) < 0 \).
Proof: Appendix.

Note that the tax-subsidy Example 1 satisfies assumption (c).

The traditional way to justify total surplus is to require that it represent each consumer’s preferences over policies. Another is to assume that aggregate demand satisfies all the properties that individual demands do, namely there is a Representative Consumer.

Assumption 1 (Representative Consumer) Aggregate demand \( \sum D^i \) maximizes a continuous, strongly monotone, strongly quasiconcave utility function \( U(X) \) subject to \( P \cdot X \leq \sum m_i = M \); and the corresponding indirect utility function \( V \) is (globally) Pareto Consistent.\(^4\)

The Representative Consumer Assumption is strong; it holds, for example, if all consumers have straight-line Engel curves with slopes common across consumers; or if the available transfers are rich enough to achieve an optimal income distribution for each price.\(^5\) The assumption nonetheless is frequently made (wittingly or not), and it is worthwhile to disentangle the question of aggregation from whether surplus maximization can pick out optimal policies for an individual. The next result shows that if the Representative Consumer Assumption holds and surplus is strongly quasiconcave, then any surplus maximizing policy is Pareto Optimal. I also give a sufficient condition for surplus to be strongly quasiconcave (part (b)).

Proposition 3 Suppose that the Representative Consumer assumption holds.

(a) If \( S \) is strongly quasiconcave, then any interior surplus-maximizing policy is Pareto Optimal.

(b) \( S \) is strongly quasiconcave (and so any interior surplus-maximizing policy is Pareto Optimal) if there is a function \( f : \text{Range}(\Phi) \to \mathbb{R}_{++} \) with \( M(\cdot) = f(\Phi(\cdot)) \) on \( A \) and, whenever \( \nabla f(\Phi(\alpha)) = D(\Phi(\alpha)) \), (i) \( \Phi'(\alpha) \neq 0 \); and (ii) the matrix \( \nabla^2 f - D_P - DD_M \) is positive definite on the subspace of prices such that \( \Phi'_i(\alpha) \neq 0 \).

Proof: Appendix.

If the economy’s production set is convex, and the Representative Consumer assumption holds, there is a unique Pareto Optimal production plan. If some policy in \( A \) achieves this plan and surplus is strongly quasiconcave then by (a) that policy also maximizes surplus. Blackorby (1999) establishes the same conclusion but with two differences: he asserts that zero income effects are necessary for the conclusion; he does not make any restrictions on the curvature of total surplus. I discuss the first point in more detail at the end of this subsection; the reason for the second difference is that I confine attention to a single price path.

The assumption in part (b) is a restriction on income effects of the policy change. Recall that the gradient of the expenditure function in price equals demand (when evaluated at appropriate income and utility levels); and the Hessian of the expenditure

\[^4\]A function \( f \) over price-income pairs is globally Pareto Consistent if \( f(P, \sum m_i) > f(P', \sum m'_i) \) implies that at least one consumer \( i \) strictly prefers \( (P, m_i) \) to \( (P', m'_i) \).

\[^5\]Mas-colell, Whinston, and Green (1995), Chapter 4.
function is the matrix $D_P + DDM$ of Slutsky substitution terms. Condition (b) implies that whenever the first-order condition for maximizing surplus is met ($\Delta f = D$), changes in income in a neighborhood of that policy do not over-compensate the representative consumer for any price changes. The condition is met by the tax-subsidy Example 1 (indeed, the condition in (b) can be used to extend Example 1 to the case in which more than one good is taxed).\(^6\)

As mentioned another, more traditional, justification is to restrict preferences and how the policy affects income so that consumer’s surplus represents each consumer’s preferences over all feasible price-income pairs, which is implied by the next two assumptions. The first one ensures that consumer’s surplus is independent of the path of integration (when the policy does not affect the consumer’s income).

**Assumption 2 (Separability)** For every $i = 1, ..., I$, there are functions $v^i$ and $\psi^i$ such that $V^i(P, m^i) = v^i(\psi^i(P), m^i)$.

The next assumption restricts the policy so that any income changes do not affect demand.

**Assumption 3 (No Income Effects)** For some positive integer $I_0 < I$ we have

(i) $m_i(\cdot)$ is constant on $[\alpha, \alpha]$ for $i = 1, ..., I_0$;

(ii) For $i = I_0 + 1, ..., I$, consumer $i$’s indirect utility function is $V^i(P, m_i) = a^i(P) + m_i$ for some function $a^i$ (so that consumer $i$’s demand for goods $1, ..., n$ are constant in income and consumer $i$ is risk neutral for money lotteries).

Part (i) is met if the income of every consumer $i \leq I_0$ consists only of goods $n + 1$ through $L$—the ones whose price relative to good $L$ is not affected by the policy (no ownership of any firm whose profit is affect by the policy and no endowment of goods 1 through $n$). This assumption is a strong one; but it is weaker than simply dividing agents into “consumers” and “producers.” Mas-cocell (1982) for example splits the economy into “capitalists” and “consumers” (see his Section 3.2): capitalists are those with ownership shares of firms, and they only care about consumption of the numeraire good. The No Income Effects assumption here is more general than the assumption in Mas-cocell (1982) since it allows capitalists to care about the consumption of all goods and it allows consumers to have ownership shares in some firms.

The next result follows immediately from Theorems 1 and 2 in Chipman and Moore (1976).\(^7\)

**Proposition 4** If the Separability and No Income Effects assumptions hold, then any surplus-maximizing policy is Pareto Optimal.

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\(^6\)In Example 1, the transfer $t(\alpha)$ plays the role of $f$ in part (b). From the calculations in the example, $t' = 0$ if and only if the tax $\alpha = 0$, and $t''(0) = 2(dp + dd_m)$, so the condition in part (b) is satisfied.

\(^7\)Chipman and Moore (1976) give conditions under which the change in consumer’s surplus represents a consumer’s preferences over price-income pairs. First, if all prices and income can move arbitrarily, the change in consumer’s surplus cannot represent a consumer’s preferences; second, if income is fixed but all price changes are allowed, then the change in consumer’s surplus represents the consumer’s preferences if and only if preferences are homothetic; finally, if the price of good $L$ is fixed but otherwise all price and income changes are allowed, then the change in consumer’s surplus represents the consumer’s preferences if and only if preferences are quasilinear with respect to good $L$. See also Chipman and Moore (1980).
My main task is to determine additional conditions under which Proposition 4 extends in the presence of uncertainty and incomplete markets.

Proposition 4 does not cover the rebated tax of Example 1. For the special case in which the policy only affects the price of good 1 and incomes, we can weaken the No Income Effects Assumption.

**Assumption 4 (Small Income Effects)** Suppose that $n = 1$ (the policy only affects the price of good 1), let part (ii) of the No Income Effects assumption hold, but replace part (i) with:

(i) For every consumer $i \leq I_0$, there is a function $f_i : \text{Range}(\phi) \to \mathbb{R}_{++}$ such that $m_i(\alpha) = f_i(\phi(\alpha))$ for all $\alpha \in A = [\underline{\alpha}, \overline{\alpha}]$ and with $f'_i(p) < d^i(p, f_i(p))$ for almost all $p$.

The Small Income Effects assumption says that changes in a consumer’s income are smaller than the Slutsky compensation for any price change; if price increases then income does not increase enough to allow the consumer to buy the pre-change consumption plan. (It ensures that an increase in the price of good 1 is bad for a consumer even if that consumer’s income changes with the policy.)

**Corollary 1** If the policy only affects the price of good 1 (i.e. $n = 1$) and the Small Income Effects assumption holds, then any surplus-maximizing policy is Pareto Optimal.

**Some Related Findings**

Facchini, Hammond, and Nakata (2001) consider a competitive production economy with a tax on a single good, and show that the Pareto Optimal tax of zero can (locally) minimize surplus. Their example clearly shows that the quasiconcavity assumption of Propositions 2 and 3 sometimes has bite and that competitive equilibria need not maximize surplus (but their example does not illustrate a surplus-maximizing policy that is not Pareto Optimal). Blackorby (1999) considers a single consumer economy. He defines surplus from a single (non-numeraire) good–good 1 say– and finds that total surplus maximization can lead the economy to a first-best equilibrium only if the demand for good 1 is independent from income. But he defines surplus (in our notation) to be $\int_\alpha^\alpha d(\phi(\theta), m(\alpha))d\theta + m(\alpha)$, rather than as we do, $\int_\alpha^\alpha d(\phi(\theta), m(\theta))d\theta + m(\alpha)$. He uses this definition of surplus to be able to identify the change in surplus using the demand function of a single good, even though the prices of many goods might be affected by the policy. The definition I use–equation (3)–is the common one. And it seems the right one to use to evaluate how total surplus is used in applications to evaluate policies. Most applications don’t explicitly include income in demand since they assume quasilinear preferences; they just specify a downward-sloping demand. If I interpret the demand function in these applications as the function $\psi(p) = d(p, f(p))$, where $f = (f_1, ..., f_{I_0})$ is the vector of functions from the small income effects assumption, then my results can be used to evaluate whether those studies identify optimal policies even in the absence of quasilinear preferences.

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8Suppose that producer prices are fixed as in Example 1 and suppose that there are two goods and two consumers. The economy’s endowment of good 2 is positive and the endowment of good 2 is zero. Suppose that consumer 2 only cares about good 2, and consumer 1’s demand (for good 1) is locally Giffen: at $p = \hat{p}$ (the producer price of good 1), consumer 1’s demand is upward sloping. The entire transfer goes to consumer 2, so that the market for good 1 is not affected by the transfer; then a small tax raises total surplus.

9For example it is the one used by Chipman and Moore (1980) and the vast literature they cite.
4 Uncertainty: Incomplete Markets

Now suppose that there is uncertainty: tastes and technologies may depend on the realization of unknown states of the world. I ask when the conclusion of Proposition 4 (or Corollary 1) extends to this case. Let $\Omega \subset \mathbb{R}$ be a set of states with typical element $\omega$ and let $F$ be a cumulative distribution function on $\Omega$. (Some applications require an infinite number of states; otherwise there is no harm of thinking of $\Omega$ as a finite set.) We assume that the policy is chosen before the state is known. After the state is realized, spot markets open for each good, so that markets are incomplete. As before, consumers take prices as given, though firms might not. I write $\phi(\alpha, \omega)$ for the equilibrium price in state $\omega$ under policy $\alpha$. From now on assume that, with probability one, the price of good 1 lies in an interval $[p, \bar{p}]$, where $0 < p < \bar{p}$. When it exists, I write the derivative of $\phi$ with to $\alpha$ in state $\omega$ as $\phi'(\alpha, \omega)$. I assume that all consumers are expected utility maximizers, and interpret the indirect utility functions as von Neumann-Morgenstern utilities over price-income lotteries: consumer $i$’s preferences over policies are now represented by the mapping $\alpha \mapsto \int V^i(\phi(\alpha, \omega), m_i(\alpha, \omega))dF(\omega)$. Part (i) of the No Income Effects Assumption—the indirect utility functions of consumers $I_0 + 1, \ldots, I$ are affine in income—now implies that these consumers are risk neutral over money lotteries.

For now I assume that the state of the world does not directly enter a consumer’s indirect utility. I can still accommodate uncertainty about consumer preferences by interpreting a consumer’s indirect utility as an interim one, after a consumer learns its type; the consumer faces price uncertainty because it doesn’t know the realized type of other consumers. (Implicit in my notation which does not index beliefs $F$ by the consumer is that consumer types are i. i. d. and each consumer is negligible.) Later I allow the state to enter the indirect utility function (Corollary 3).

In many applications, consumers’ attitudes towards price risk is irrelevant for the positive predictions of the model: at the time the consumers choose, all the price uncertainty is resolved. (This is true of all the examples discussed in Section 5.) Such attitudes are relevant of course for evaluating policies that must be chosen before any uncertainty is resolved.

4.1 Pareto Inconsistency: an example

Suppose that the policy only affects the price of good 1 and that the policy has no effect on any consumer’s income. (For example, profit could be zero for each policy choice and in each state.) In addition, suppose that all consumers have identical preferences which are quasilinear with respect to good $L$, but that all consumers are risk averse over money lotteries. Let consumer $i$’s vN-M indirect utility be

$$V^i(p, m_i) = T_i \left( \int_p^{\bar{p}} d(\theta) d\theta + m_i - \gamma_i \right).$$

for some positive real number $\gamma_i$. In particular, set $\gamma_i = m_i$.

Suppose now that the random price of good 1 becomes riskier in the Rothschild-Stiglitz sense as $\alpha$ increases. As an example, the technology for good 1 could be constant.

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The Representative Consumer approach of Theorem 3 assumption is problematic when there is uncertainty and incomplete markets, since the Representative Consumer can easily violate Pareto Consistency even if all consumers have straight-line Engel curves with common slope: the Representative Consumer can prefer policy $A$ over $B$ even if every consumer prefers $B$ to $A$ (e.g. Schlee (2001)).
returns with unit cost $k(\alpha, \omega)$ in state $\omega$, and markets are competitive for each $(\alpha, \omega)$; the assumption is that productivity shocks become more volatile with an increase in $\alpha$.\footnote{This example is reminiscent of calculating the welfare cost of business cycles, as in Lucas (1987), but with a simple static production economy, rather than a dynamic exchange economy.} Since consumer surplus is a strictly convex function of price, expected surplus of each consumer is strictly increasing in $\alpha$ (Waugh (1944)). Let $\epsilon_i$ be the supremum of $-d'(\phi(\alpha, \omega))/d(\phi(\alpha, \omega))^2$ on $A \times \Omega$. If $T_i$ is any twice differentiable, strictly increasing and concave function satisfying

$$-T''_i(z)/T'_i(z) > \epsilon_i \quad \text{for all} \quad z \in \mathbb{R}_+,$$

then each $V^i$ is strictly concave in $p$, and each consumer $i$ is made worse off with an increase in the riskiness of price. This example shows that any policy recommendation which relies on the familiar fact that expected consumer's surplus increases with price risk will not be extend to economies with sufficiently risk averse consumers.

### 4.2 Local Pareto (In)consistency

As the example in Section 4.1 shows, expected aggregate surplus is not in general even locally Pareto consistent. Local Pareto consistency fails since the uncertainty analogue of equation (2) generally fails. For this subsection, let $A$ be an interval containing the point 0 in its interior. Differentiate expected indirect utility with respect to the policy to find (using Roy’s Identity and assuming that price path is smooth enough so that we can interchange differentiation and expectation)

$$d \overline{E}[V^i(\Phi(\alpha, \omega), m_i(\alpha, \omega))] = E \left[ (-D^i\Phi' + m'_i) V^i_{m} \right]. \quad (6)$$

Unless the marginal utility of income $V^i_{m}$ does not depend on prices, (6) is not in general the same sign as the derivative of expected consumer’s surplus,

$$E \left[ (-D^i\Phi' + m'_i) \right]. \quad (7)$$

Of course, if (7) is positive, then consumer $i$’s expected surplus is locally increasing in some states $\omega$. In that weak sense, expected total surplus is locally Pareto consistent. But as the example in Section 4.2 shows, it can still happen that every consumer ranks policies the opposite way that expected total surplus does.

There is however one way to restore local Pareto consistency without any preference restriction (except differentiability of the vN-M utility). Suppose that at policy $\alpha = 0$, there is no price or income uncertainty: $(\phi(0, \omega), m_i(0, \omega)) = (\hat{p}, \hat{m}_i)$ for all $\omega$. Evaluate (6) at $\alpha = 0$ to find that (assuming that derivative and integral can be interchanged)

$$d \overline{E}[V^i(\Phi(\alpha, \omega), m_i(\alpha, \omega))]_{\alpha=0} = \left( -D^iE[\Phi'(0, \omega)] + E[m'_i(0, \omega)] \right) V^i_{m}(\hat{P}, \hat{m}_i), \quad (8)$$

so expected consumer’s surplus locally represents each consumer’s policy preferences in a neighborhood of $\alpha = 0$, provided that the right side of (8) is nonzero.
Proposition 5 Suppose that the random prices and incomes are smooth enough in $\alpha$ to allow interchange of the derivative and expectation, and that there is no price or income uncertainty at $\alpha = 0$.\textsuperscript{12} If the derivative of aggregate expected surplus with respect to $\alpha$ is not $0$ at $\alpha = 0$, then expected total surplus is locally Pareto consistent at $\alpha = 0$.

The assumption that there is no uncertainty at $\alpha = 0$ is of course a strong one. It’s conceivable that there is no price uncertainty at 0 if price is set by a regulator, or that firms must set price before any uncertainty is resolved (as in Section 5.3 on Information Acquisition). More surprisingly, the smoothness assumption justifying the interchange of derivative and integral in equation (8) is not just a technical assumption, but rules out some cases found in the literature (Schlee (2008), Section 4). For example, suppose that a policy has no income effects and that the price of good 1 equals $p_0$ with probability $1-\alpha$, and is either $p_1$ or $p_2$ with probability $\alpha/2$ each, with $p_0 = (p_1 + p_2)/2$, so that an increase in $\alpha$ results in a mean-preserving increase in risk, as in Section 4.2. In particular, $\phi(\alpha, \omega)$ equals $p_1$ if $\omega \in (0, \alpha/2)$, equals $p_1$ if $\omega \in (1 - \alpha/2, 1]$ and equals $p_0$ otherwise. Obviously, there is no uncertainty at $\alpha = 0$. It is easy to show that $\phi(\cdot, \omega)$ is differentiable at $\alpha = 0$ for every $\omega$ (the derivative equals zero). But expected total surplus is not locally Pareto consistent at $\alpha = 0$: $dE[S(\alpha, \omega)]/d\alpha > 0$ at $\alpha = 0$, but if each consumer is sufficiently risk averse, $E[V^i(\phi(\alpha, \omega), m_i)]$ is strictly decreasing in $\alpha$.

4.3 Optimality and Expected Surplus Maximization

Besides the No Income Effects Assumption, the main theorem on the optimality of expected-surplus-maximizing policies imposes two restrictions, one on preferences, the other on how the policy affects prices. The preference restriction is that each consumer’s indirect utility function is supermodular in price and income: equivalently, the marginal utility of income $V^i_m(p, m)$ is increasing in $p$.\textsuperscript{13} An (twice continuously differentiable) indirect utility function is supermodular in income and the price of good 1 if an only if the implied relative risk aversion for money lotteries is at least as large as the budget elasticity for good 1.\textsuperscript{14} Intuitively, supermodularity implies that risk aversion is more important than income effects on demand. Budget elasticities, on average across goods, must equal 1 simply from budget balance. Estimates of relative risk aversion over money lotteries vary, but they almost alway exceeds 1, with many much higher. Meyer and Meyer (2005) summarize the empirical evidence. The estimates differ in part because some studies take the outcome of the money lottery to be wealth, and others

\textsuperscript{12}One assumption that justifies the interchange of derivative and expectation is that the difference quotient $q(\alpha, \omega) - q(\bar{\alpha}, \bar{\omega})/\bar{\alpha} - \alpha$ is bounded uniformly in $(\alpha, \omega)$ on $A \times \Omega$.

\textsuperscript{13}Let $f$ be a real-valued function on $X \times T$, where $X$ and $T$ are subsets of the real line. $f$ is supermodular (SPM) if for every $\bar{x}, x$ in $X$ and $\bar{t}, t$ in $T$

$$f(\max\{\bar{x}, x\}, \max\{\bar{t}, t\}) + f(\min\{\bar{x}, x\}, \min\{\bar{t}, t\}) \geq f(\bar{x}, \bar{t}) + f(x, t).$$

If $X$ and $T$ are intervals and $f$ is twice continuously differentiable, then supermodularity is equivalent to $f_{xt} \geq 0$ on $X \times T$, where subscripts denote partial derivatives.

\textsuperscript{14}To prove the equivalence, differentiate Roy’s Identity $d = -V_p/V_m$ with respect to income, multiply both sides by $m/d$ and rearrange to find $\varepsilon_m = -mV/m + r$, where $r$ is relative risk aversion and $\varepsilon_m$ is the income elasticity of demand for good 1.
Figure 1: Single-crossing property: policy $\beta$ leads to prices which are less variable than prices from policy $\alpha$.

Take it to be income or consumption. If it is wealth, estimates typically range between 1 and 4; if it is consumption or income, estimates are usually much higher.\textsuperscript{15} Although budget elasticities vary across goods, surely on average they are less than relative risk aversion for money lotteries, so supermodularity of the indirect utility is surely the typical case.

The policy restriction is that expected surplus maximizing policies generate random prices that are less variable than other policies in the sense that, if $\alpha^*$ maximizes expected surplus, then the random prices from other policies cross $\phi(\alpha^*, \cdot)$ at most once from below (Figure 1).\textsuperscript{16}

Suppose that the No Income Effects–Uncertainty holds. For $i = 1, \ldots, I_0$ define consumer $i$’s surplus in state $\omega$ to be

$$S_i(\alpha, \omega) = \int_{\phi(\alpha, \omega)}^{\beta} d^i(\theta, m_i) d\theta$$

and expected surplus for consumer $i$ to be

$$E[S_i(\alpha, \omega)] = \int_{\Omega} S^i(\alpha, \omega) dF(\omega).$$

For consumers $I_0 + 1, \ldots, I$ let $E[S_i(\alpha, \omega)] = \int_{\Omega} \left[ \int_{\phi(\alpha, \omega)}^{\beta} d^i(\theta) d\theta + m_i(\alpha, \omega) \right] dF(\omega)$ be the expected consumer surplus. Set $E[S(\alpha, \omega)] = \sum_{i=1}^{I} E[S_i(\alpha, \omega)]$.

**Theorem 1** Let $\alpha^* \in A$ and suppose that consumer $i$ strictly prefers $\alpha \in A$ to $\alpha^*$. If

\textsuperscript{15}In more recent work, Cohen and Einav (2007) use information about insurance demand to estimate absolute and relative risk aversion with income as the lottery outcome. The estimates vary greatly across individuals depending on what contract they choose; the the mean individual has relative risk aversion equal to 97, the median, only 0.37. As I explain later, the assumption that every consumer have a supermodular indirect utility can be substantially relaxed.

\textsuperscript{16}As an example, the mean-output preserving decreases in price risk in Newberry and Stiglitz (1979) satisfy this condition.
(i) the No Income Effects Assumption–Uncertainty holds;
(ii) the policy only affects the price of good 1;
(iii) \( V^i \) is increasing in the price of good 1; and
(iv) \( \phi(\cdot, \alpha^*) \) crosses \( \phi(\cdot, \alpha) \) at most once from above;

then \( E[S_i(\alpha^*, \omega)] < E[S_i(\alpha, \omega)] \). Hence if \( \alpha^* \) maximizes expected surplus on \( A \), condition (iv) holds for every \( \alpha \in A \), and condition (iii) holds for every consumer \( i \), then \( \alpha^* \) is Pareto Optimal.

Proof: Appendix.

The idea of the proof can be explained using equation (2), \( E[dV] = E[V_m dS] \). Under the single-crossing condition, \( dS < 0 \) when price is low, and \( dS > 0 \) when price is high. Since \( V_m \) is increasing in price, the function \( V_m \) puts more weight on states in which \( dS \) is positive. It follows that \( E[V_m dS] > \gamma E[dS] \) for some positive number \( \gamma \), from which the conclusion follows.

4.3.1 Extensions and remarks

1. Theorem 1 shows more than the optimality of policies which maximize expected surplus; it shows that if everyone prefers \( \alpha \) to \( \alpha^* \), (iii) holds for every consumer \( i \leq I_0 \), and (iv) holds, then expected surplus is higher under \( \alpha \) than \( \alpha^* \). Expected surplus is a ‘one-way’ representation of the Pareto ordering of the two policies (namely each consumer strictly prefers \( A \) to \( B \) only if expected surplus is higher for \( A \) than \( B \)).

2. Although I argue that supermodularity of the indirect utility for every consumer is a mild assumption, it can be weakened: if \( \alpha^* \in A \) maximizes expected surplus, then for every policy \( \alpha \in A - \alpha^* \), there is some consumer \( i \) with a supermodular indirect utility whose expected surplus is higher under \( \alpha^* \) than \( \alpha \).

3. The No Income Effects Assumption can be relaxed somewhat for consumers 1 through \( I_0 \).

Assumption 5 (The Small Income Effects Assumption–Uncertainty) \( A = [\underline{\alpha}, \overline{\alpha}] \); the equilibrium price and each consumer’s income is differentiable on \( A \) for every \( \omega \in \Omega \); part (ii) of the No Income Effects assumption holds, but part (i) is replaced with:

(i) For every \( i \leq I_0 \), \( \alpha \in A \) and \( \omega \in \Omega \), there is a function \( f_i : [\underline{p}, \overline{p}] \to \mathbb{R}_{++} \) such that \( m_i(\alpha, \omega) = f_i(\phi(\alpha, \omega)) \) on \( A \times \Omega \), with

\[
f'_i(\phi(\alpha, \omega)) \leq d(\phi(\alpha, \omega), f_i(\phi(\alpha, \omega))) \left( 1 - \frac{\varepsilon_i(\alpha, \omega)}{r_i(\alpha, \omega)} \right)
\]

for every \( (\alpha, \omega) \in A \times \Omega \) where \( \varepsilon_i(\alpha, \omega) \) is consumer \( i \)'s income elasticity of demand and \( r_i(\alpha, \omega) \) consumer \( i \)'s relative risk aversion at \( (\phi(\alpha, \omega), m_i(\alpha, \omega)) \).

If a good is normal for a risk-averse consumer, then condition (i) is stronger than the No Income Effects assumption for that consumer; but in the limit as risk aversion increases without bound, the two conditions are the same. I prove the following Corollary (and its successor) in the Appendix.

\[\text{Let } P \text{ be a binary relation on a set } X. \text{ A real-valued function } f \text{ on } X \text{ is a one-way representation of } P \text{ if, for any } x, y \in X, xPy \text{ implies } f(x) > f(y). \text{ The terminology is found in Aumann (1961).}\]
Corollary 2  The conclusion of Theorem 1 holds if the No Income Affects Assumption is replaced with the Small Income Effects Assumption–Uncertainty.

The small income effects assumption ensures that each consumer’s marginal utility of income still increases in price, even when income is price-dependent; informally, the argument given after the statement of Theorem 1 that \( E[dV] > \gamma E[dS] \), goes through.

4. We can extend Theorem 1 to the case of state-dependent utility, provided that the state and income are additively separable in the indirect utility function. This extension is important if uncertainty affects consumer tastes, but the policy is chosen before learning the realization of the event which determines their tastes.

Corollary 3  The conclusions of Theorem 1 hold if each consumer’s indirect utility function takes the form \( V^i(p,m,\omega) = g^i(p,\omega) + h^i(p,m) \) with \( h^i \) supermodular.

Schlee (2007), Section 3.4, has an extended discussion of the meaning of this form of additive separability. Under risk aversion (\( h^i \) concave in \( m \)) it is a special form of “increasing dispersion:” for two types \( \omega_0 \) and \( \omega_1 \) of consumer \( i \) with the same income, if type \( \omega_1 \) buys more of good one than type \( \omega_0 \), then type \( \omega_1 \)'s marginal propensity to consume good 1 out of income is higher than type \( \omega_0 \)'s; in other words, the income-expansion paths of the two types become more dispersed as income rises.

Additive separability implies that the marginal utility of income does not depend on the state. The conclusion of Corollary 3 would still hold with additive separability if it so happened that equilibrium prices \( \phi(\alpha,\cdot) \) ordered the states the same way for every \( \alpha \in A \); and the marginal utility of income were higher in states in which prices were higher.

4.3.2  Example: commodity price stabilization

I illustrate Theorem 1 with a classic model of commodity price stabilization.\(^{18}\) Suppose that all firms are price takers and that producer prices for all goods other than good 1 do not vary with the state. Let \( d(p,\omega) \) be the aggregate demand for good 1 and \( s(p,\omega) \) the aggregate supply of good 1 in state \( \omega \in \Omega \subset [0,1] \). The government owns a positive buffer stock of good 1 and decides on net sale \( a(\omega) \) of the stock in each state \( \omega \in \Omega \). Denote the equilibrium price of good 1 in state \( \omega \) under (measurable) state-contingent sales \( a(\cdot) \) by \( \phi_a(\omega) \): \( d(\phi_a(\omega),\omega) = s(\phi_a(\omega),\omega) + a(\omega) \) and expected total surplus is

\[
\int_\Omega \int_0^\infty d(p,\omega)d\phi_a(\omega)dF + \int_\Omega \int_0^{\phi_a(\omega)} s(p,\omega)d\phi_a(\omega)dF + \int_\Omega a(\omega)d\phi_a(\omega)dF.
\]

Suppose that the constraint facing the government is that the stock on average should not change: \( \int_\Omega a(\omega)dF = 0 \). It is easy to show that any solution to this problem requires complete price stabilization: \( \phi_a(\omega) = \text{constant} \) for (almost all) states. Now fix any measurable function \( a(\cdot) \) and let \( a^* \) maximizes expected total surplus over all feasible functions. Identify policy \( \alpha = 1 \) with \( a^* \) and \( \alpha = 0 \) with \( a \). If consumers have supermodular indirect utility functions which are additively separable in the state and income, then by Theorem 1 or Corollary 3, \( \alpha = 1 \) is still Pareto Optimal on \( A = \{0,1\} \).

\(^{18}\)See for example, Newberry and Stiglitz (1979) and their references.
4.4 When does Theorem 1 apply?

In the next section I ask whether policies which maximize expected total surplus are Pareto Optimal in several important applications if consumers have supermodular indirect utility functions. A striking conclusion is that answer is “yes” for the models with demand-side uncertainty, and generally “no” for models with supply-side uncertainty. I now address why this conclusion is plausible.

Theorem 1 requires that the expected surplus maximizing policy generate a random price that crosses the price from any other policy at most once from above; in that sense the random price from the expected surplus maximizing policy is least variable of all available random prices. In some applications these two conditions can be checked from the characterization of the equilibrium of the model. But two obvious questions are, first, in general what economies generate prices that are ordered by the single-crossing property; and when will the least variable of these random prices maximize expected total surplus? In short, when does Theorem 1 apply?

The next result provides a partial answer to the second question: if the state only affects demand, the prices are ordered by the single-crossing property, and the least variable price has the lowest mean, then under one additional condition the policy with the least variable price of good 1 maximizes expected consumer surplus.\(^{19}\)

Hence if expected income of consumers \(I_0, ..., I\) (the owners of firm 1, all of whom have quasilinear utility) is constant across policies, the least variable price also maximizes expected total surplus. The additional condition is that, for each policy, equilibrium output and price move in the same direction in response to different state realizations, that is, equilibrium output and price are \(\text{comonotonic}\) in the state of the world:

\[
[d(\phi(\alpha, \omega'), \omega') - d(\phi(\alpha, \omega), \omega)] \times [\phi(\alpha, \omega') - \phi(\alpha, \omega)] \geq 0 \text{ for any } \omega, \omega' \text{ in } \Omega.
\]

**Theorem 2** Let the set of policies \(A\) be an interval. If the state only affects demand, the equilibrium output \(d(\phi(\alpha, \cdot), \cdot)\) and price \(\phi(\alpha, \cdot)\) are \(\text{comonotonic}\) for all policies \(\alpha \in A\), the random prices from any two policies satisfy the single crossing property, and the mean of the least variable price is no larger than the mean price from any other policy, then the least variable price policy maximizes expected consumer surplus. Hence if the No Income Effects–Uncertainty assumption holds and the mean income of each consumer \(i > I_0\) is the same for all policies, then the policy with the least variable price maximizes expected total surplus.

**Proof:** Appendix.

Intuitively, when equilibrium price and quantity demanded are \(\text{comonotonic}\), then expected consumers’ surplus rises with a mean-preserving (or mean-reducing) contraction, since price falls when quantity demanded is high and possibly rises only when quantity demanded is low. On the other hand if the state only affects supply, and the other assumptions of Theorem 2 hold–except that the mean of the price from the \(\text{more variable}\) policy is no larger than that from any other policy–then the policy with the most variable price maximizes expected consumers’s surplus (since consumer surplus is convex in price). See Figure 2. Hence if expected income of consumers \(i > I_0\) is constant across policies, the most variable price also maximizes expected total surplus.

\(^{19}\)As I discuss in more detail in what follows, Theorem 2 generalizes Theorem 4 in Deneckere, Marvel, and Peck (1997) to a much broader class of models.
Figure 2: Both panels depict a mean-preserving contraction in state prices. The left shows a fixed demand, the right a variable demand in which demand and price are comonotonic. With a mean preserving contraction in the price distribution, expected consumers’ surplus falls in the left panel, but rises in the right panel. 

(as in Section 4.1). Intuitively, when equilibrium price and minus quantity demanded are comonotonic, then expected consumers’ surplus rises with a mean-preserving (or mean-reducing) spread, since price rises when quantity demanded is high and (possibly) falls when quantity demanded is low.

The answer the first question–what policy changes yield random prices which are ordered by the single-crossing property–depends both on the model and the set of allowed policies. But even at a high level of generality, the set of policies considered might also depend on what the source of uncertainty in the model is. If the state only affects demand, then the price of the good will be high in states that consumers value it highly; it would be natural to consider policy changes which increase output in high demand states (possibly by decreasing output in low-demand states)—a change that would generate less variable prices. If the state only affects the technology, then price is high in states in which the inputs used to produce the good are less productive; it would be natural in that case to consider policy changes which increase output in states with low marginal cost (possibly by decreasing output in states with high marginal cost)—a change which would generate more variable prices.

To formalize this idea, I suppose that there is a real-valued function \( y \) on \( \mathbb{R}_+ \times A \times \Omega \) such that the equilibrium price of good 1 at policy \( \alpha \) and state \( \omega \) is the solution \( p \) to

\[
y(p, \alpha, \omega) = d(p, \omega).
\]

I call \( y(\cdot) \) an output function. For example, if firms are price takers, then the output function \( y(\cdot, \alpha, \omega) \) can be taken to be the supply function of good 1. An output function, however, generally exists even if firms are not competitive.\(^\text{20}\) For example, suppose that the firm is a regulated monopolist with private information about \( \omega \). Suppose that a regulator seeks to maximize a weighted sum of expected consumers’ surplus and profit, and suppose the comparison is between the regulator’s chosen policy \( \alpha_1 \) and that of an unregulated monopolist \( \alpha_0 \). We can take \( p = \phi(\alpha_1, \omega) \) to be the price and \( y(\phi(\alpha_1, \omega), \alpha_1, \omega) \) that the regulator requires when the firm reports its type \( \omega \); and

\(^{20}\)There can easily be more than one output function for a market.
Theorem 3. Let β ∈ A and let α₀ be any other element of A. Order the states in Ω by \( \phi(\beta, \omega) \) and define β to be a higher policy than α₀. Suppose that there is an output function y which satisfies the following conditions.

1. It is supermodular in \((p, \alpha)\) on \([p, p] \times \{\alpha_0, \beta\}\).
2. It is supermodular in \((\omega, \alpha)\) on \(\Omega \times \{\alpha_0, \beta\}\).
3. The excess demand \(z\) is strictly decreasing in \(p\).

Then \(\phi(\beta, \cdot)\) crosses \(\phi(\alpha_0, \cdot)\) at most once from above.

Proof: Appendix.

Intuitively, Theorem 3 says that if output under the higher policy β is more sensitive to price and more sensitive to the state (which are ordered by prices under policy β) then the price under policy β crosses the price under the lower policy α₀ at most once from above.

Broadly, Theorems 2 and 3 suggest that if consumers have supermodular indirect utilities, the uncertainty arises on the consumer side of the economy, and the policies order the equilibrium output by how sensitive it is to the state of the world and to price—as already mentioned a natural modeling choice when the state only affects demand—then expected surplus maximizing policies will be Pareto Optimal. I illustrate with an example of a constant-returns economy with consumer-side uncertainty which contrasts with the constant-returns economy with producer-side uncertainty of Section 4.1.

Example 2 (Constant returns, uncertain demand) Consider first a two-good economy with a strictly convex, strictly monotone, constant-returns technology which produces good 1 using both goods 1 and 2 (Cobb-Douglas, for example). Consumers 1 through \(I_0\) are not endowed with any good 1, but can own firms; consumers \(I_0 + 1\) through \(I\) are endowed with a positive amount of good 1 (and possibly firm shares and good 2). Aggregate demand depends on the state of the world \(\omega\), but the technology does not. Since the set of feasible final consumption plans is generally strictly convex, equilibrium prices generally will depend on the state of the world. Suppose that is the case and let \(\phi(\omega)\) be an equilibrium price in state \(\omega\). Now consider a second economy identical to the first except that the technology is no longer strictly convex, but linear.
with production function given by \( f(z_1, z_2) = kz_1 + z_2 \), where \( z_j \) is the amount of good \( j \) used as an input in the production of good 1. In particular set \( k = E[\phi(\omega)] \), so the price of good 1 in the second economy is a mean-preserving contraction of the random price in the first economy. In particular, the hypotheses of Theorem 2 or Corollary 3 are met (in particular the mean income of consumers \( I_0 + 1, ..., I \) is the same in the two economies). Two conclusions follow: expected total surplus is higher with the second technology; and the choice of the second technology over the first is a Pareto Optimal policy.

5 Applications

As mentioned, the single crossing condition is often met in applications of expected surplus. I consider four applications here, all from Industrial Organization. The applications illustrate the argument that the hypotheses of Theorem 1 are more likely to be met if the uncertainty is about demand, rather than supply.

5.1 Minimum Resale Price Maintenance (RPM)

Deneckere, Marvel, and Peck (1997) propose a theory of ruinous price competition under demand uncertainty.\(^{21}\) A good is produced by a monopolist manufacturer and sold by a continuum of identical retailers. Retailers must order inventories before the demand uncertainty is resolved and any unsold inventories are worthless. They consider two scenarios. The first is flexible pricing: the manufacturer sets a wholesale per-unit price, \( p_w \), that each retailer pays at the time that inventories are ordered. After the demand uncertainty is resolved, the retail price is set so that demand equals aggregate inventory. The second scenario is minimum resale price maintenance (RPM): the manufacturer sets both a wholesale price \( p_w \) and a minimum price, \( p_{\text{min}} \), below which the retail price cannot fall. If demand equals supply at a price above \( p_{\text{min}} \), then the market-clearing price prevails; otherwise the retail price is set at \( p_{\text{min}} \) and consumers are rationed among retailers to equalize the ex ante probability of a sale.

Strikingly, they find that expected total surplus can easily be higher under RPM than under flexible pricing (see their Theorem 3 on page 632 and Theorem A1 on page 638)—hence ‘ruinous’ price competition. RPM can even raise expected consumers’ surplus (Theorem 4). Intuitively, minimum RPM leads retailers to order more inventory (Theorem 2); the increased output means a lower retail price when demand is high. The trade-off for consumers is between higher prices when demand is low and lower prices when demand is high. As this intuition suggests, prices are more variable under flexible pricing than under minimum RPM. We reproduce their Figure 1 (see Figure 3), which gives the supply for flexible pricing and minimum RPM: the intersection of the realized demand with supply determines the retail price. If demand is low, the flexible price is lower and if demand is high, it is higher than the price from minimum RPM: the random price from RPM crosses the flexible price once from above. By Theorem 1, if the No Income Effects Assumption holds and each consumer’s relative risk aversion is higher than its income elasticity of demand for the good, the conclusion

\(^{21}\)See also Deneckere, Marvel, and Peck (1996).
Figure 3: Figure 1 from Deneckere, Marvel, and Peck (1997): The intersection of the realized demand with the output function determines the price; the output function for flexible pricing is \( y(p, FL) \) and for minimum RPM it is \( y(p, RPM) \).

of Deneckere, Marvel, and Peck (1997)’s Theorems 3 and 4 continue to hold: minimum RPM is Pareto Optimal.

Deneckere, Marvel, and Peck (1997)’s Theorem 4 shows that if expected price under minimum RPM is not larger than the expected price under flexible pricing, then expected consumers’ surplus is higher under minimum RPM. My Theorem 2 extends this result to a much broader class of environments and clarifies that the important forces are that prices satisfy the single-crossing property; and that equilibrium demand and price are comontonic.

5.2 Regulating a Monopolist with Private Information

Consider a regulator and a firm who produces a single good at a single date. The firm has private information either about its cost or its demand. The regulator’s objective is to maximize a weighted sum of expected consumers’ surplus and expected profit, with more weight on consumers’s surplus.

5.2.1 Cost Uncertainty

Baron and Myerson (1982) consider a firm whose production technology exhibits non-decreasing returns to scale with constant marginal costs (whenever output is positive). The firm’s cost function is private information to the firm, but both the firm and regulator know the firm’s demand. The firm reports a cost type to the regulator and the regulator offers a menu of type-contingent prices, quantities and transfers between consumers and the firm to maximize expected total surplus subject to incentive compatibility and participation constraints for the firm. As Baron and Myerson point out, the surplus-maximizing policy may involve setting prices for some cost types which are higher than the highest possible monopoly price in the absence of regulation. Compare the menu which maximizes expected total surplus with one generated by an unregulated monopoly (so transfer payments are zero). Clearly the sufficient condition of Theorem 1 fails here. Indeed, since the worst outcome for consumers can occur under the menu which maximizes expected total surplus, it can easily happen that all
consumers and the firm can prefer no regulation at all to the policy which maximizes expected total surplus.\footnote{Consider zero fix-cost case with demand such that an unregulated firm always produces. Suppose regulated firm shuts down for some cost types. In case case the worst outcome for consumers occurs under regulation, since the transfer from consumers is zero (equation (28) on page 920) when the firm shuts down. The same phenomenon can also occur if the firm does not shut down: \( \bar{p}(\theta) > p_M(\theta_1) \geq p_M(\theta) \) for some \( \theta \), but the firm stays in business.}

5.2.2 Demand Uncertainty

Lewis and Sappington (1988) consider the case in which the cost is known to both the regulator and firm, but the demand function is private information to the firm. They show that, if the cost function is strictly concave, then the policy which maximizes total surplus (subject to incentive and participation constraints) is to set a price which is constant in the demand report of the firm (Proposition 2).

If the cost function is strictly convex, the optimal regulatory policy is the first best (for every cost type price is equal to marginal cost). Obviously this is Pareto Optimal even if consumers do not have quasilinear utility. If the cost function is concave, then the regulator sets the same price for every report and the firm is held to zero profit for every report. If the no or small income effects assumption (for the uncertainty case) holds, and each consumer’s indirect utility function is spm, then Theorem 1 implies that this policy is Pareto Optimal.

5.3 Information Acquisition

Suppose firms can acquire information about either demand or cost. Does maximizing expected total surplus result in Pareto Optimal information acquisition? Vives (1999), Section 8.3.3 considers a constant-returns monopolist which is uncertain about the intercept of its linear demand curve. Before it chooses either price or quantity, it can choose the informativeness (precision) of a signal correlated with a demand parameter, with more informative signals costing more.\footnote{Vives (1999) uses the monopoly case to introduce the larger issue of information acquisition and sharing in oligopoly, a question going back to Novshek and Sonnenschein. It’s clear that if expected total surplus cannot lead to optimal policies for a monopoly market it will not in general do so for an oligopoly market.} Vives (1999) shows that if the monopolist is a quantity setter (letting price be determined by the realized demand) expected consumer surplus rises with a more informative signal, so expected total surplus obviously rises as well. Since the firm produces more after a good signal, and less after a bad, than it would with less information, prices become less variable with better information. By Theorem 1, choosing a more to a less informative signal is Pareto Optimal if consumers have supermodular indirect utilities. But he shows that if the monopolist is a price setter, both expected total and expected consumers’ surplus fall with better information, so a planner interested in expected surplus, but who cannot control the monopolist’s pricing decision would prefer zero information to be acquired. Since information acquisition makes prices more variable, this conclusion again fits Theorem 1: every consumer with a supermodular indirect utility agrees that no information is the best policy. What if the uncertainty is about cost instead of demand? Schlee (2008), Section 4.1, shows that if uncertainty is about cost, expected consumers’ surplus rises with better information for a price (or quantity) setter, but if consumers are sufficiently
risk averse, all consumers prefer the less informative signal. If information and costly and the firm is just indifferent between acquiring it and not, then the Pareto optimal decision would be not to acquire it, even though expected total surplus is higher with information.

If the planner can control output of the firm the planner now always prefers more information to less. Persico (1996), Section 6.1, considers information about unknown demand and Athey and Levin (2001), Section 4.1, consider information about unknown cost. They each find that under mild conditions that the firm acquires less information than would a planner whose objective is expected total surplus. Let the inverse demand function be \( p(q, \omega) \) and cost function be \( c(q, \omega) \). A planner whose objective is total surplus sets output so that expected price equals expected marginal cost: \( E[p(q, \omega)] = E[c'(q, \omega)] \). If only cost is unknown, it is clear that information can make consumers with supermodular indirect utilities worse off: with known demand, price varies only when output varies, and information makes output vary more; indeed if information is costly, it can happen that no one prefers and some consumers are strictly worse off with better information, even though expected total surplus rises. Suppose instead that cost is known but demand is uncertain. Price now becomes less variable with better information since the planner adjusts output in the same direction as demand after a signal is observed (provided the mild stability condition \( p' - c'' < 0 \) holds). Summing up, suppose for concreteness that the planner chooses between no information and some information and suppose that the planner chooses output to maximize expected surplus (so the planner always prefers some information to none if it is costless). If demand is uncertain, then the planner’s information choice is Pareto Optimal; if cost is uncertain the planner’s information choice might not be.

As a final example, Bradford and Kelejian (1977) considers a two-period model in which the period-2 production is subject to exogenous random fluctuations (as in agriculture). A group of agents “speculators,” can buy period-1 production to sell during period 2. They find that expected total surplus rises with better information but that expected consumer surplus falls with better information–expected consumers’ surplus falls precisely because prices undergo a mean-preserving decrease in risk with better information, so a consumer with a supermodular indirect utility function could prefer more information.

### 5.4 Price Caps under Uncertainty

Earle, Schmedders, and Tatur (2007) consider a Cournot oligopoly model with identical firms producing under constant returns to scale and uncertain demand. There is a legal maximum price that firms may charge—the ceiling price. The timing of production and resolution of demand uncertainty is the same as Deneckere, Marvel, and Peck (1997). Firms choose production before demand is realized; after demand is realized price adjusts to the lower of two numbers, the market-clearing price and the ceiling price. They give sufficient conditions under which, contrary to the certainty case, an increase in the ceiling price can raise equilibrium output, expected total surplus and expected consumer surplus. In this case, raising the ceiling price makes price even higher when demand is high, but even lower when demand is low: raising a price cap makes the random price more variable, not less. Their model is outside the scope of Theorems 1-3, which assume market clearing: consumers are able to consume all that they demand at a given price. With price caps and production in advance of the demand realization,
consumers are rationed. Nonetheless it is easy to see that their main conclusion—raising price caps can raise welfare when demand is uncertain—easily extends beyond the quasilinear utility. If the price cap is set at the unit cost of production, then a monopolist would produce the competitive output for the lowest demand realization. If this output is zero, then a price cap set at unit cost causes the firm to shut down altogether. If the price cap is removed altogether, and the monopolist produce some output in some states of the world, then no consumer is hurt by removing the price cap, and some of them are strictly better off (as is the firm). If we interpret shut-down as charging a high enough price so that demand is zero in each state, raising the price cap in this example results in a first-order stochastic dominance shift in the price distribution.

6 Conclusion

Economists often recommend policies which maximize total surplus. One advantage of the surplus approach is that it does not require the economist to specify details of consumer preferences and endowments. Sometimes this lack of detail might be desirable since many different preference/endowment specifications can lead to the same demand. Moreover, with no uncertainty and complete markets, it often finds policies which are Pareto Optimal even if consumer preferences over price-income pairs are not globally represented by consumer’s surplus; this happy outcome occurs because consumer preferences over price-income pairs are at least locally represented by consumer’s surplus.

When there is uncertainty and incomplete markets, expected consumer’s surplus does not even locally represent a consumer’s preferences over (random) price-income pairs. Nonetheless I identify conditions under which policies which maximize expected surplus are Pareto Optimal. The preference restriction is just that each consumer’s indirect utility function is supermodular, equivalently, each consumer’s coefficient of relative risk aversion over money lotteries is at least as large as the income elasticity of demand for the good whose price is affected by the policy, roughly that risk aversion over money lotteries is more important than income effects on the demand for the good whose price is affected by the policy.

7 Appendix: Proofs

The proofs appear in the same order as the results in the text.

7.1 Proof of Proposition 2

Clearly, (b) follows from (a). For part (a), suppose that \( S'_i(\alpha^*) > 0 \) for some consumer \( i \). If \( \alpha^* = 1 \), then it is Pareto Optimal, since any small decrease in \( \alpha \) hurts consumer \( i \). If \( \alpha^* < 1 \), then for some consumer \( j \) we must have \( S'_j(\alpha^*) < 0 \) (since \( S'(\alpha^*) \leq 0 \) for \( \alpha^* < 1 \)). Then a small increase in \( \alpha \) hurts consumer \( j \) and a small decrease hurts consumer \( i \), so \( \alpha^* \) is locally Pareto Optimal. The argument for \( S'_i(\alpha^*) < 0 \) is similar.

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\(^{24}\) Einav, Finkelstein, and Cullen (2008) make this argument in the context of their model of adverse selection in insurance markets.
For part (c), suppose that $S''(\alpha^*) < 0$. By part (a), it suffices to consider the case in which $S'_i(\alpha^*) = 0$ for all consumers $i$, which of course implies that $\frac{d}{d\alpha} V^i(\Phi(\alpha), m_i(\alpha)) = 0$ at $\alpha^*$. Consider the additive social welfare function $W(\alpha) = \sum \lambda_i V^i$ with weights given by $\lambda_i = 1/V^i_m(\Phi(\alpha^*), m^i)$ for each $i$. Obviously $W''(\alpha^*) = 0$. We show that $W'''(\alpha^*) < 0$, implying that $\alpha^*$ is a local maximizer of $W$, hence it is locally Pareto Optimal. We have

$$W'''(\alpha^*) = \sum \left[ \lambda_i \mu_i(\alpha^*) S''_i(\alpha^*) + \lambda_i S'_i(\alpha^*) \mu'_i(\alpha^*) \right] = S'''(\alpha^*) < 0. \quad (12)$$

So $\alpha^*$ is locally Pareto Optimal. ■

7.2 Proof of Proposition 3

(a) If $S$ is strongly quasiconcave, then so is $W(\alpha) = V(\Phi(\alpha), m(\alpha))$. The first-order conditions–$S'_i(\alpha) = 0$ for $\alpha \in (\underline{\alpha}, \overline{\alpha})$, $S'_i(\alpha) \leq 0$ if $\alpha = (\underline{\alpha}$ and $S'_i(\alpha) \geq 0$ if $\alpha = \overline{\alpha}$–are both necessary and sufficient for maximizing $S$. We have $W'(\alpha) = V_m(\Phi(\alpha), m(\alpha)) S'_i(\alpha)$, so if $\alpha^*$ maximizes $S$, it also maximizes $W$. (b) Whenever $S'(\alpha) = \Phi'(\alpha)(\nabla f(\Phi(\alpha)) - D(\Phi(\alpha))) = 0$, we have

$$W''(\alpha) = \Phi'(\alpha)^T \times [\nabla f''(\Phi(\alpha)) - D_P(\Phi(\alpha)) - D(\Phi(\alpha)) D_M(\Phi(\alpha))] \times \Phi'(\alpha)$$

where we have shortened the price-income argument $(\Phi(\alpha), f(\Phi(\alpha)))$ in the demands to just $\Phi(\alpha)$. (c) If $S'(\alpha) \neq 0$ for every $\alpha \in (\underline{\alpha}, \overline{\alpha})$, then both $S$ and $W$ are either strictly increasing or strictly decreasing, so the conclusion holds. If $S'(\hat{\alpha}) = 0$ for some $\hat{\alpha} \in (\underline{\alpha}, \overline{\alpha})$, then $W'(\hat{\alpha}) = 0$. Moreover $W''(\hat{\alpha}) = S''(\hat{\alpha})$. Since $W$ is strongly quasiconcave $W''(\hat{\alpha}) < 0$, so it follows that $\hat{\alpha} = \alpha^*$; and $\hat{\alpha}$ locally maximizes $S$. Since $S(\alpha)' \neq 0$ for every $\alpha \in A - \{\hat{\alpha}\}$, $\hat{\alpha}$ is the unique global maximizer of $S$ on $A$. ■

7.3 Proof of Theorem 1

Let the hypotheses of the Theorem hold. The first conclusion is obvious for any consumer $i > I_0$, so fix a consumer $i \leq I_0$. There is an increasing real valued function $T$ such that

$$T \left( \int_p^p d(\theta, m_i) d\theta + m_i \right) = V^i(p, m_i)$$

for all $p \in [p, \overline{p}]$, since surplus and $V^i$ are both strictly decreasing in $p$. Differentiate both sides wrt $p$ and use Roy’s Identity to find that $T'(s_i(p)) = V^i_m(p, m_i)$, where $s_i(p)$ is the argument of $T$, consumer $i$’s surplus at (the nonrandom) price $p$. Since $V^i_m$ is increasing, and $s_i$ is decreasing, in $p$, it follows that $T$ is concave. The conclusion now follows from standard comparative statics arguments. Fix $\alpha \in A$, and let $\Delta S_i(\omega) = S_i(\alpha^*, \omega) - S_i(\alpha, \omega)$ and $\Delta T(\omega) = T(S_i(\alpha^*, \omega)) - T(S_i(\alpha, \omega))$. Since $T$ is concave, $\phi(\cdot, \alpha^*)$ crosses $\phi(\cdot, \alpha)$ at most once from above, and surplus is decreasing in price, there is a positive number $R$ such that $\Delta T(\omega)/\Delta S(\omega) \leq R$ if and only if $\Delta S(\omega) < 0$. 

23
So

\[
\int \left( V^i(\phi(\alpha^*, \omega), m_i) - V^i(\phi(\alpha, \omega), m_i) \right) dF(\omega)
\]

\[= \int \Delta T(\omega) dF(\omega) = \int \frac{\Delta T(\omega)}{\Delta S_i(\omega)} \Delta S(\omega) dF(\omega) \]

\[= \int \left( \frac{\Delta T(\omega)}{\Delta S_i(\omega)} - R \right) \Delta S_i(\omega) dF(\omega) + R \int \Delta S_i(\omega) dF(\omega) \geq 0 + R \int \Delta S_i(\omega) dF(\omega) \]

\[= R\left( E[S_i(\alpha^*, \omega)] - E[S_i(\alpha, \omega)] \right). \tag{13} \]

where the inequality follows since the first integrand is nonnegative everywhere. Since \( R > 0 \), equation (13) implies that if consumer \( i \) strictly prefers \( \alpha \) to \( \alpha^* \), then \( i \)'s expected surplus is higher for \( \alpha \) than \( \alpha^* \), so if (iii) holds for every consumer \( i \leq I_0 \), and every consumer prefers \( \alpha \) to \( \alpha^* \), \( \alpha^* \) cannot maximize surplus. It follows that if (i)-(ii) hold, \( \alpha^* \) does maximize surplus, (iv) holds for every \( \alpha \in A \), and (iii) holds for every consumer \( i \leq I_0 \), then \( \alpha^* \) is Pareto Optimal. ■

7.4 Proof of Corollary 2

Let the hypotheses of the Corollary hold and fix a consumer \( i \leq I_0 \), and from now on suppress the consumer superscript \( i \). Define

\[ s(p) = \int_p^b \{d(\theta, f(\theta))d\theta + f(p) \}
\]

for every \( p \) in the range of \( \phi(\alpha, \cdot) \) or in the range of \( \phi(\alpha^*, \cdot) \). By the Small Income Effects Assumption, \( s(\cdot) \) is strictly decreasing in \( p \) and so has an inverse \( s^{-1} \) which is strictly decreasing. Define a real-valued function \( T \) on the range of \( s \) by

\[ T(s(p)) = V(p, f(p)) \tag{14} \]

for all \( p \) in the domain of \( s \). Since \( s(p) \) and \( V(p, f(p)) \) are strictly decreasing in \( p \), such a function exists and is strictly increasing. Fix \( \omega \in \Omega \), insert \( \phi(\alpha, \omega) \) into the \( p \) argument on both sides of (14), and differentiate it with respect to \( \alpha \) to find that \( T'' = [V_{mp} + V_{mm}f'] ds^{-1}/dz \leq 0 \). The rest of the proof follows the proof of Theorem 1. ■

7.5 Proof of Corollary 3

Fix a consumer \( i \leq I_0 \) and from now on suppress the consumer superscript, and let \( V(p, m, \omega) = g(p, \omega) + h(p, m) \). For each \( \omega \in \Omega \), there is strictly increasing function \( T(\cdot, \omega) \) such that \( V(p, m, \omega) = T(s(p), \omega) \). Differentiate with respect to \( p \) to find that \( T_s(s, \omega) = g_m \). Since \( g \) does not depend on \( \omega \), it follows that \( T \) is additively separable in \( s \) and \( \omega \); and since \( V \) is supermodular, it follows that \( T \) is concave in its first argument. The rest of the proof follows the proof of Theorem 1. (That \( T \) is additively separable in \( s \) and \( \omega \) is critical here.) ■
7.6 Proof of Theorem 2

Let \( \alpha \) in \( A \) order the policies from least variable to most and let \( \beta \) be the smallest (least variable) policy. For any \( \alpha^* \in A \) we have that the difference in expected surplus between policy \( \beta \) and \( \alpha \) is

\[
\int_{\Omega} \left[ \int_{\phi(\beta, \omega)}^{\phi(\alpha, \omega)} d(p, \omega)dp \right] dF(\omega)
\]

\[
= \int_{\Omega} \int_{\beta}^{\alpha^*} \frac{d}{d\alpha} \left[ \int_{\phi(\alpha, \omega)}^{\phi(\alpha^*, \omega)} d(p, \omega)dp \right] d\alpha dF(\omega)
\]

\[
= \int_{\Omega} \int_{\beta}^{\alpha^*} \left[ -d(\phi(\alpha, \omega), \omega)\phi'(\alpha, \omega) \right] d\alpha dF(\omega)
\]

\[
= \int_{\Omega} \int_{\beta}^{\alpha^*} \left[ -\int \phi(\alpha, \omega)\phi'(\alpha, \omega) dF(\omega) \right] \tag{15}
\]

Since \( \alpha \) orders the prices by the single crossing property, for each \( \alpha \in [\beta, \alpha^*] \) there is a \( \omega_\alpha \in \Omega \) such that \( \phi'(\alpha, \omega) \leq 0 \) for almost all \( \omega < \omega_\alpha \) and \( \phi'(\alpha, \omega) \geq 0 \) for almost all \( \omega \geq \omega_\alpha \). Define \( f(\alpha, \omega) = d(\phi(\alpha, \omega), \omega) - d(\phi(\alpha, \omega), \omega_\alpha) \), which for each \( \alpha \in [\beta, \alpha^*] \) is increasing in \( \omega \) and is of the same sign as \( \phi'(\alpha, \omega) \). Since \( \int_{\Omega} \phi'(\alpha, \omega)dF(\omega) = 0 \) (the mean prices are the same) it follows that the last term in (15) is

\[
= \int_{\Omega} \int_{\beta}^{\alpha^*} \left[ -\int f(\alpha, \omega)\phi'(\alpha, \omega) dF(\omega) \right] \leq 0,
\]

so by (15) and (16), \( \beta \) maximizes expected surplus. \( \blacksquare \)

7.7 Proof of Theorem 3

Let \( \phi(\alpha_0, \omega) \geq \phi(\beta, \omega) \) and \( \phi(\beta, \omega') \geq \phi(\beta, \omega) \). To verify the single-crossing condition I need to show that \( \phi(\alpha_0, \omega') \geq \phi(\beta, \omega') \), and since \( z \) is strictly decreasing in \( p \) for that it suffices to show that \( z(\phi(\beta, \omega'), \alpha_0, \omega') \geq 0 \). By equation (11), \( 0 = z(\phi(\beta, \omega), \beta, \omega) \).

Add and subtract terms of the form \( z(\phi(\tilde{\alpha}, \tilde{\omega}), \tilde{\alpha}, \tilde{\omega}) \) find that

\[
0 = B + C + z(\phi(\beta, \omega), \alpha_0, \omega) - z(\phi(\beta, \omega'), \alpha, \omega') \tag{17}
\]

where \( B = \) \( z(\phi(\beta, \omega), \beta, \omega) - z(\phi(\beta, \omega'), \beta, \omega') \) \( - \left( z(\phi(\beta, \omega), \alpha_0, \omega) - z(\phi(\beta, \omega'), \alpha_0, \omega) \right) \)

\[
= \left( y(\phi(\beta, \omega'), \beta, \omega) - y(\phi(\beta, \omega), \beta, \omega) \right) - \left( y(\phi(\beta, \omega'), \alpha_0, \omega) - y(\phi(\beta, \omega), \alpha_0, \omega) \right)
\]

and

\[
C = \left( z(\phi(\beta, \omega'), \beta, \omega) - z(\phi(\beta, \omega'), \beta, \omega') \right) - \left( z(\phi(\beta, \omega'), \alpha_0, \omega) - z(\phi(\beta, \omega'), \alpha_0, \omega') \right)
\]

\[
= \left( y(\phi(\beta, \omega'), \beta, \omega') - y(\phi(\beta, \omega'), \beta, \omega') \right) - \left( y(\phi(\beta, \omega'), \alpha_0, \omega') - y(\phi(\beta, \omega'), \alpha_0, \omega) \right)
\]

\[
= \left( y(\phi(\beta, \omega'), \beta, \omega') - y(\phi(\beta, \omega'), \beta, \omega') \right) - \left( y(\phi(\beta, \omega'), \alpha_0, \omega') - y(\phi(\beta, \omega'), \alpha_0, \omega) \right)
\]

\[
= \left( y(\phi(\beta, \omega'), \beta, \omega') - y(\phi(\beta, \omega'), \beta, \omega') \right) - \left( y(\phi(\beta, \omega'), \alpha_0, \omega') - y(\phi(\beta, \omega'), \alpha_0, \omega) \right)
\]
The second equality in equations (18) and (19) each follow since demand does not directly depend on the policy. Since $y$ is supermodular in $(\alpha, p)$ and $\phi(\beta, \omega') \geq \phi(\beta, \omega)$, it follows that $B \geq 0$; since $y$ is supermodular in $(\alpha, \omega)$ (with the states ordered by $\phi(\beta, \omega)$ and with $\beta$ higher than $\alpha_0$) and $\phi(\beta, \omega') \geq \phi(\beta, \omega)$, it follows that $C \geq 0$; and since $z$ is strictly decreasing in $p$ and $\phi(\alpha_0, \omega) \geq \phi(\beta, \omega)$, it follows that $z(\phi(\beta, \omega), \alpha_0, \omega) \geq 0$. Combine these last three inequalities to conclude by (17) that $z(\phi(\beta, \omega), \alpha_0, \omega) \leq 0$, and so that $\phi(\alpha_0, \omega') \geq \phi(\beta, \omega')$.

References


